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An Analysis of Spin Diffusion Dominated Ferrofluid Spin-up Flows in Uniform Rotating Magnetic Fields

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Abstract: This work analyzes the spin-diffusion dominated mechanism for spin-up bulk flows in ferrofluid filled cylinders, with no free surface, subjected to a uniform rotating magnetic field. COMSOL results are compared to experimental and analytical results computed using Mathematica, giving good agreement between the three. Both one (ferrofluid cylinder) and two domain (ferrofluid cylinder in air) problems solved in this work are compared and yield identical resulting in further insight into the governing physics. Subtleties occur when using a single domain region to model the ferrofluid cylinder without correct adjustment for demagnetizing effects. COMSOL was also used to investigate spin viscosity effects on velocity profiles. Overall, this work documents the method of simulating ferrofluid spin-up flows, in uniform rotating magnetic fields, using COMSOL Multiphysics. The COMSOL results obtained give good agreement to analytical solutions and experimental results.

Keywords: ferrofluids, uniform rotating magnetic fields, spin-up flows, spin diffusion, ferrohydrodynamics

1. Introduction

The classic spin-up experiment involves placing a ferrofluid filled cylinder in a uniform rotating magnetic field and observing the velocity distribution. The opacity of the ferrofluid led many researchers to observe the velocity distribution using streak path techniques with tracer particles only on the surface of the fluid [1-3]. However this led to observations of flow that were counter-rotating to the rotational direction of the magnetic field [4]. This counter-rotating phenomena was explained to be a result of asymmetric tangential stresses on the boundary of the magnetic fluid [5]. It was believed that the flow in the bulk of the fluid would be an entrainment of this surface flow. If asymmetric tangential stresses on the boundary of the magnetic fluid entrained the fluid layers below, then by placing a cover and removing the free surface at the top of the cylinder the fluid would conceivably not have any motion [3, 6-9].

Pulsed ultrasound velocimetry allows for bulk velocity flow measurements of opaque fluids and experiments were done with and without the cover on top of the surface. Co-rotating motion was observed in the bulk of the fluid in both cases while counter-rotating motion, with a concave shaped meniscus, was observed near the surface [10-13].

There are two alternate theories for bulk ferrofluid spin-up flow in rotating fields. One theory considers inhomogeneous heating of the fluid due to the dissipated energy of the rotating field to create a spatial variation in susceptibility driving the rotational flow [3, 14-16]. Another theory assumes that the imposed rotating magnetic field is non-uniform itself due to the demagnetizing effects associated with a finite height cylinder [3, 15]. Recent work has demonstrated that this non-uniform magnetic field drives ferrofluid flow [17-21].

This work analyzes the spin diffusion theory explanation for rotational motion by assuming the cylinder's height is infinite, resulting in negligible field non-uniformity in most of the fluid, and assumes that the strength of the magnetic field is weak enough to assume negligible heating effects of the fluid.

2. Governing Equations

2.1 Ferrohydrodynamics

The governing ferrohydrodynamics equations include conservation of linear and angular momentum [22] given in Eqs (1) and (2).

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=-\nabla p'+2\zeta\nabla\times\boldsymbol{\omega}+(\zeta+\eta)\nabla^2\mathbf{v}+\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} \tag{1}$$

$$I\left[\frac{\partial \boldsymbol{\omega}}{\partial t}+(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}\right]=\mu_0\mathbf{M}\times\mathbf{H}+2\,\zeta(\nabla\times\mathbf{v}-2\boldsymbol{\omega})+\eta'\nabla^2\boldsymbol{\omega} \tag{2}$$

where the variables are dynamic pressure p' (including gravity) [N/m^2], ferrofluid mass density ρ [kg/m^3], fluid moment of inertia density I [kg/m], fluid magnetization $\mathbf{M}$ [A/m], magnetic field $\mathbf{H}$ [A/m], spin velocity $\boldsymbol{\omega}$ [1/s], ferrofluid dynamic viscosity η [N-s/m^2], vortex viscosity $\zeta=1.5\eta\varphi_{vol}$ [N-s/m^2] for small volume fraction φ_{vol} of magnetic nanoparticles [22, 23], and η' [N-s] is the shear coefficient of spin viscosity.

2.2 Magnetic Field Equations

The ferrofluid magnetization relaxation equation derived by Shliomis [23] is

$$\frac{\partial \mathbf{M}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{M} = \boldsymbol{\omega} \times \mathbf{M} - \frac{1}{\tau_{\text{eff}}}(\mathbf{M} - \mathbf{M}_{\text{eq}}) \tag{3}$$

where equilibrium magnetization $\mathbf{M}_{\text{eq}}$ [A/m] is given by the Langevin equation

$$\mathbf{M}_{\text{eq}} = M_s\left[\coth(\beta) - \frac{1}{\beta}\right]\left(\frac{\mathbf{H}}{H}\right), \beta = \frac{M_d V_p \mu_0 H}{kT} \tag{4}$$

with M_s [A/m] the saturation magnetization given as, $M_s=M_d\varphi_{vol}$, M_d = 446 [kA/m] is the domain magnetization for magnetite [22], V_p [m^3] is the magnetic core volume per particle, $\mu_0=4\pi\times10^{-7}$ [H/m] is the magnetic permeability of free space, k=1.38x10^{-23} [J/K] is Boltzmann's constant, T [K] the temperature in Kelvin, and effective relaxation time constant τ_{eff} [s] includes Brownian and Néel effects.

Maxwell's equations for a non-conducting fluid are

$$\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} = 0, \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{5}$$

The magnetic field can be given by

$$H = -\nabla\psi \tag{6}$$

where ψ is the magnetic scalar potential. Eqs (5) and (6) result in a Poisson's equation given as

$$\nabla^2\psi = \nabla \cdot \mathbf{M} \tag{7}$$

3. Assumptions

The applied field is assumed to not be strong enough to magnetically saturate the fluid. The equilibrium magnetization $\mathbf{M}_{\text{eq}}$ [A/m] of the fluid is assumed to be in the linear regime of the Langevin equation. This linear relationship as a function of the magnetic field *inside the ferrofluid* has a slope represented by χ, the magnetic susceptibility of the material, is given by

$$\mathbf{M}_{\text{eq}} = \chi \mathbf{H}_{\text{fluid}} \tag{8}$$

An infinitely long ferrofluid filled cylinder was modeled since it has no demagnetizing effect in the axial direction and equal demagnetizing factors of ½ in the transverse directions resulting in a uniform field inside the ferrofluid filled cylinder. In the actual experiments, the finite height cylinder of ferrofluid does have an internal non-uniform field, due to demagnetizing effects, in an external uniform field.

The flow is also assumed to be incompressible and dominated by viscous effects (low Reynolds number) allowing for the inertial terms to be dropped. The left hand side of the linear and angular momentum equations in Eqs (1) and (2) can be set to 0 reducing the equations to

$$\mathbf{0} = -\nabla p' + 2\zeta\nabla \times \boldsymbol{\omega} + (\zeta + \eta)\nabla^2\mathbf{v} + \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} \tag{9}$$

$$\mathbf{0} = \mu_0\mathbf{M} \times \mathbf{H} + 2\,\zeta(\nabla \times \mathbf{v} - 2\boldsymbol{\omega}) + \eta'\nabla^2\boldsymbol{\omega} \tag{10}$$

If the magnetic field is applied in the transverse x-y plane, the spin-velocity $\boldsymbol{\omega}$ is assumed to only be in the z-direction ω_z. This is because in an infinitely long cylinder, the driving force is created only by the transverse magnetic field which creates a torque only in the z-direction. The spatially varying demagnetizing field of a finite height cylinder would create an internal magnetic field that had components in the transverse (x-y) plane as well as the axial plane (z). In that case, there would be a torque and spin-velocity $\boldsymbol{\omega}$ in all three directions (x, y and z).

4. Theory

The theoretical steady state description of the ferrofluid entrainment in a rotating magnetic field is done in the cylindrical coordinate system where the fluid velocity only has an azimuthal component (v_φ) and the spin velocity is only in the vertical z direction (ω_z) with rotating magnetic field components in the Cartesian x and y directions.

Analytical spin diffusion expressions for the azimuthal velocity and spin velocity profiles have previously been derived [22, 24] and are given in Eqs 11-12.

$$\omega_z(r) = \frac{\zeta + \eta}{\eta(R)}\left(\frac{\mu_0|\mathrm{M}||\mathrm{H}_{\mathrm{fluid}}|}{4\zeta}\sin\alpha\right)\left[1 - \frac{I_0(\kappa r)}{I_0(\kappa R)}\right] \quad \textbf{(11)}$$

$$\mathrm{v}_\varphi(r) = v_0\left[\frac{r}{R} - \frac{I_1(\kappa r)}{I_1(\kappa R)}\right] \quad \textbf{(12)}$$

where

$$\eta(R) = \eta + \zeta\left[1 - \frac{2I_1(\kappa R)}{\kappa R I_0(\kappa R)}\right] \quad \textbf{(13)}$$

$$v_0 = \frac{1}{2\kappa\eta(R)}(\mu_0|\mathrm{M}||\mathrm{H}_{\mathrm{fluid}}|\sin\alpha)\frac{I_1(\kappa R)}{I_0(\kappa R)} \quad \textbf{(14)}$$

$$\kappa^2 = \frac{4\eta\zeta}{(\zeta + \eta)\eta'} \quad \textbf{(15)}$$

The ferrofluid magnetization **M**, is obtained as a solution from the relaxation equation given in Eq. 3. I_0 and I_1 represents the modified Bessel function of the first kind of order zero and one respectively.

The angle α represents the lag angle between the magnetization vector **M** and the applied rotating magnetic field $\mathbf{H}_{applied}$ rotating at angular frequency Ω. It can be determined to be a solution to a cubic algebraic equation given in Eq. 16 [22]. Eq. 16 is derived assuming that there is bulk flow is absent in the fluid [22]. The effect of absent bulk flow on the lag angle α is too small to result in any considerable effect on the solutions of Eq. 11 & 12.

$$x^3 - (\Omega\tau_{\mathrm{eff}})x^2 + (P+1)x - \Omega\tau_{\mathrm{eff}} = 0 \quad \textbf{(16)}$$

where

$$P = \frac{\mu_0 \mathrm{M}_{\mathrm{eq}}\mathrm{H}_{\mathrm{fluid}}\tau_{\mathrm{eff}}}{4\zeta} \quad \textbf{(17)}$$

and

$$x = \tan\alpha \quad \textbf{(18)}$$

A subtlety is the adjustment of the magnetic field applied outside the ferrofluid to the magnetic field inside the ferrofluid. For a rotating magnetic field applied in the Cartesian x-y plane this adjustment is given by Eqs 19 and 20, where ½ represents the demagnetizing factors for an infinitely long cylinder with a field being applied transverse to the cylinder's axis. $\mathrm{H}_{\mathrm{applied}_\mathrm{x}}, \mathrm{H}_{\mathrm{applied}_\mathrm{y}}$ represent the applied rotating magnetic field amplitude in the x and y directions, $\mathrm{H}_{\mathrm{fluid}_\mathrm{x}}, \mathrm{H}_{\mathrm{fluid}_\mathrm{y}}$ represent the magnetic field internal to the ferrofluid and M_x and M_y represent the ferrofluid magnetization in x and y directions.

$$\mathrm{H}_{\mathrm{fluid}_\mathrm{x}} = \mathrm{H}_{\mathrm{applied}_\mathrm{x}} - \frac{1}{2}\mathrm{M}_\mathrm{x} \quad \textbf{(19)}$$

$$\mathrm{H}_{\mathrm{fluid}_\mathrm{y}} = \mathrm{H}_{\mathrm{applied}_\mathrm{y}} - \frac{1}{2}\mathrm{M}_\mathrm{y} \quad \textbf{(20)}$$

5. Modeling

5.1 Model Setup and Parameters

A spin-up experiment was modeled using parameters taken for ferrofluid EMG900_2 used by Chaves in [12]. The parameters used are documented in Table 1.

5.2 Using Mathematica

Eqs 11-18 were solved using Mathematica 8.0 to give analytical solutions for the spin-up problem using parameters taken in Table 1.

Parameter	Value
τ_{eff} (s)	$1\mathrm{x}10^{-6}$
ρ (kg/m^3)	1030
η (Ns /m^2)	0.0045
$\mu_0 M_s$(mT)	23.9
ζ (Ns/m^2)	0.0003
Frequency (Hz)	85
Radius of cylindrical vessel (m)	0.0247
Radius of stator (m)	0.0318
Volume Fraction (%)	4.3
χ	1.19
Ω (rad/s)	534.071
η' (kg m/s)	$6\mathrm{x}10^{-10}$
B_0 (mT) *RMS*	10.3,12.5, 14.3
B_0 (mT) *amplitude*	14.57,17.68, 20.22

Table 1. Table of physical and experimental parameters used by Chaves in one spin-up experiment using EMG900_2.

5.3 Using COMSOL Multiphysics 3.5a

5.3.1 Modeling the Rotating Magnetic field

There are two ways to model the magnetic field, one is by using a current source boundary condition (AC/DC module – Perpendicular Induction Currents, Vector Potential) while the other is by using a scalar potential boundary condition (General PDE). Both methods were explored using parameters taken from Table 1.

5.3.2 Surface Current Boundary Condition

The actual experimental setup is similar to Figure 1 and involves placing a cylinder of ferrofluid inside a stator winding (itself surrounded by a material with assumed infinite magnetic permeability) of radius R_0 with an air region in between. The resulting magnetic field in the air region is a uniform field imposed by the current source and a dipole field created by the ferrofluid.

The source of the magnetic field is a 3 phase 2 pole stator winding with each phase having 120° phase difference from each other. This requires a surface current boundary condition driving the three phase coils in the axial (z) direction of the cylinder which can be described by Eq. 21 where Ω is the rotational frequency and φ the angle from the x axis.

$$\mathbf{K}(\varphi, t) = \frac{3}{2} K_0 \cos\left(\Omega t - \varphi\right) \boldsymbol{i_z} \tag{21}$$

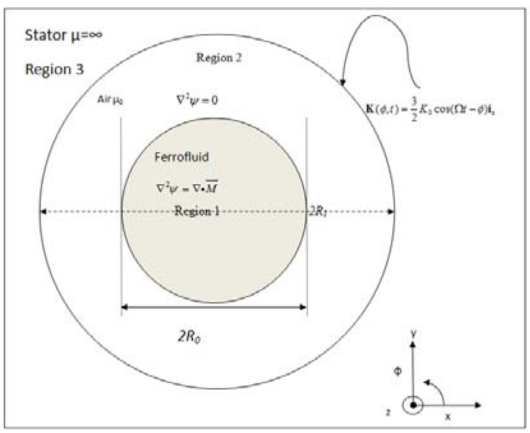

Figure 1. Two dimensional representation of actual spin-up flow experiment. Shaded region represents the infinitely long cylinder of ferrofluid of radius R_0. The unshaded air region separates the ferrofluid from the outer stator winding that has a current boundary condition imposed at $r=R_1$ surrounded by a μ=∞ region.

5.3.3 Scalar Potential Boundary Condition

Although setting up the model using the current boundary condition can be accomplished in COMSOL Multiphysics, it can be difficult to solve. A simpler method of setting up the magnetic field using the magnetic scalar potential can be used and has been used before by other authors [25]. The fact that the ferrofluid region of interest is only affected by the uniform field imposed allows for this setup to be simplified to a one region problem, similar to Figure 2, to aid the numerical simulation process.

To describe the uniform rotating field in Cartesian coordinates the external fields ($\mathbf{H}_{\text{applied}}$) are sinusoidal functions of time with rotational frequency Ω and 90^0 out of phase with each other. Eq. 22 generates a counter-clockwise uniform rotating magnetic field of magnitude H_0 and the corresponding magnetic scalar potential is given in Eq. 23.

$$\mathbf{H}_{\text{applied}} = H_0\left(\cos(\Omega t)\, \boldsymbol{i_x} + \sin(\Omega t)\, \boldsymbol{i_y}\right) \tag{22}$$

$$\psi = H_0(x\cos(\Omega t) + y\sin(\Omega t)) \tag{23}$$

where H_0 is related to the surface current as

$$H_0 = \frac{3}{2} K_0 \tag{24}$$

The one region case has a subtlety which was explained in section 3 where a correction of the field value used *inside* the ferrofluid ($\mathbf{H}_{\text{fluid}}$) has to be made as seen in Eqs 19 & 20. This is then the field source substituted in Eq. 8 and used in the magnetic relaxation equation in Eq. 3.

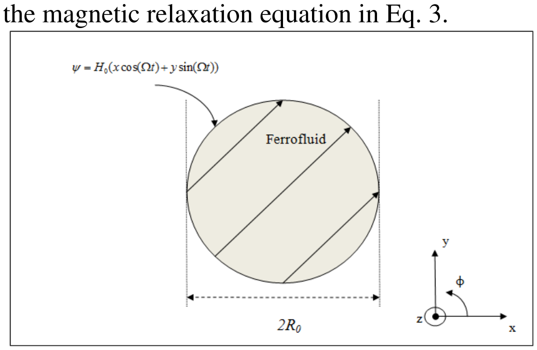

Figure 2. One region model setup with shaded circle representing ferrofluid with linear magnetization and boundary condition on magnetic scalar potential. The scalar potential generates a magnetic field rotating in the φ direction at frequency Ω. This magnetic field represents the external magnetic field and has to be corrected for demagnetizing effects before being used in the magnetic relaxation equation. The arrows inside the stator show the uniformly distributed rotating magnetic field created inside the ferrofluid at a particular instant in time.

5.3.4 Modules used and Fluid Boundary Conditions

All equations were put into COMSOL in non-dimensional form and in all cases the transient form of the equations were used.

The fluid mechanics module was used to represent the augmented Navier-Stokes equation in Eq. 9 using the no slip velocity boundary condition at the wall boundary given as

$$\mathbf{v}(r = R_0) = 0 \quad \mathbf{(25)}$$

Two transient convection and diffusion modules were used to represent the magnetic relaxation equation, in the x and y coordinate systems, in Eq. 3.

A diffusion equation was used for the angular momentum equation in Eq. 10. The spin diffusion condition ($\eta' \neq 0$) uses the "no spin slip" boundary condition which assumes that the ferrofluid nanoparticle/wall interactions are so strong that there is no relative spin between the nanoparticles and the wall boundary.

$$\boldsymbol{\omega}(r = R_0) = 0 \quad \mathbf{(26)}$$

If $\eta'=0$, there is no boundary condition on the spin velocity $\boldsymbol{\omega}$ and the resulting flow velocity is 0.

6. Results

6.1 Comparing Experimental Results, Analytical Results using Mathematica and COMSOL 3.5a Results

Experimental results obtained by Chaves in [12] gave good agreement with analytical solutions obtained using Mathematica and finite element solutions using COMSOL 3.5a as seen in Figure 4.

The COMSOL solutions were obtained using either the scalar potential or surface current boundary condition using an extremely fine mesh.

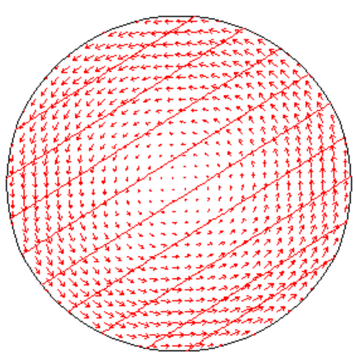

Figure 3. Velocity flow profile and magnetic field distribution using either the scalar potential or surface current boundary condition method for a counter-clockwise rotating field. Flow velocity is depicted by arrows counter-rotating while streamlines depict the uniform rotating magnetic field distribution in the region of ferrofluid.

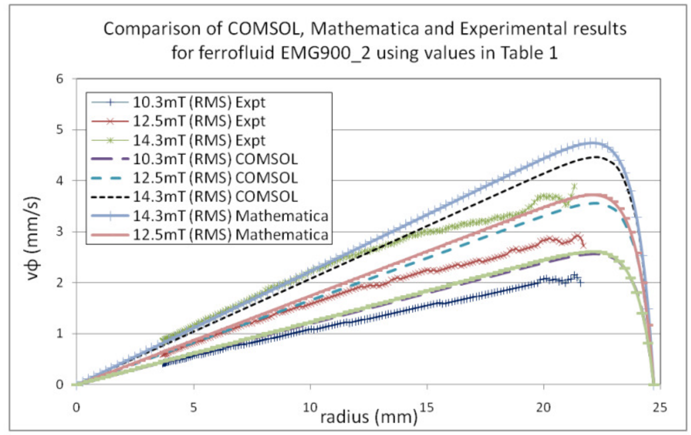

Figure 4. Comparison of COMSOL, Mathematica and experimental results by Chaves [12] for ferrofluid (EMG900_2) spin-up flow results obtained using three different magnetic field strengths and parameters in Table 1.

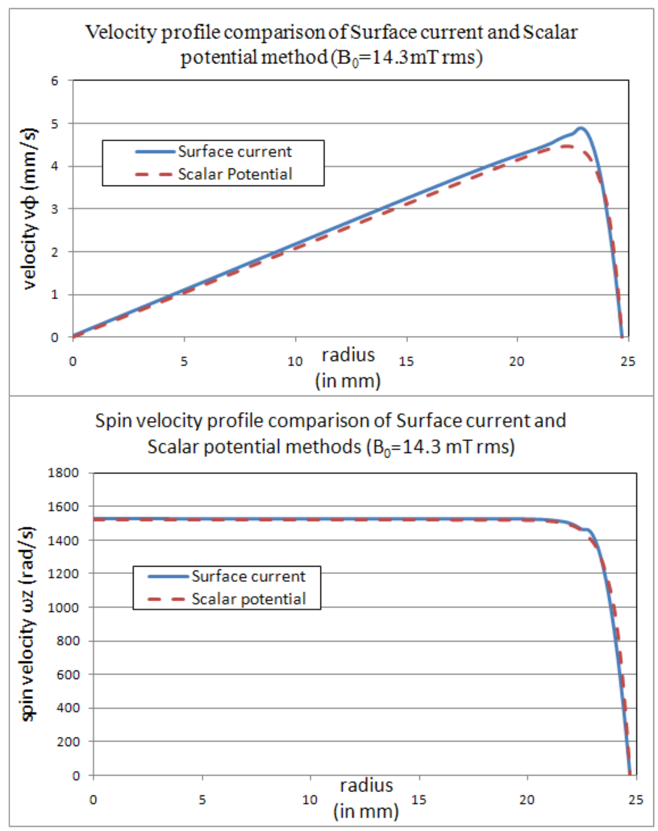

Figure 5. Plot of rotational velocity and spin velocity as a function of radius comparing the two different implementations on source boundary conditions - Surface current and scalar potential. Profiles can be seen to be almost identical in both implementations.

6.2 Comparison of Surface Current and Scalar Potential Boundary Condition Methods using COMSOL 3.5A

Rotating flow that co-rotated with the field direction were obtained similar to Figure 3 when using parameters in Table 1 for all three different field strengths.

Figure 5 shows two plots comparing both boundary condition methods for the velocity and spin velocity profiles. The surface current method has a slight peak due to numerical error associated with meshing a larger subdomain. Apart from the slight numerical error, both methods can be seen to be identical with the rotational velocity being linear with radius except for at the boundary where a no slip boundary condition has to be satisfied. The spin velocity is constant within most of the fluid region going to zero to satisfy the "spin no-slip" boundary condition.

7. Discussion

The value of the implementing the two different boundary conditions for ferrofluid spin-up flows helps to give further insight into the underlying physics.

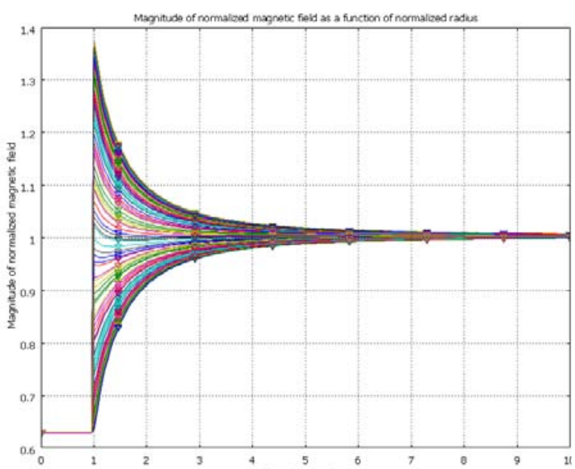

Figure 6. Magnitude of normalized magnetic field distribution as a function of normalized radius inside and outside the cylinder of ferrofluid. The different colors represent the magnetic fields at various times. It is evident that the field inside the cylinder is uniform and outside the cylinder it is a uniform and a line dipole field, that decays as the cube of the distance, generated outside. The stator is excited at a normalized radius (R_l=10) sufficiently far away from the ferrofluid. Magnitude of the normalized magnetic field inside the ferrofluid is approximately 0.629.

When using the surface current boundary condition, the field distribution inside (uniform) and outside the ferrofluid cylinder (uniform + dipole field) can be seen in Figure 6 (the values are normalized).

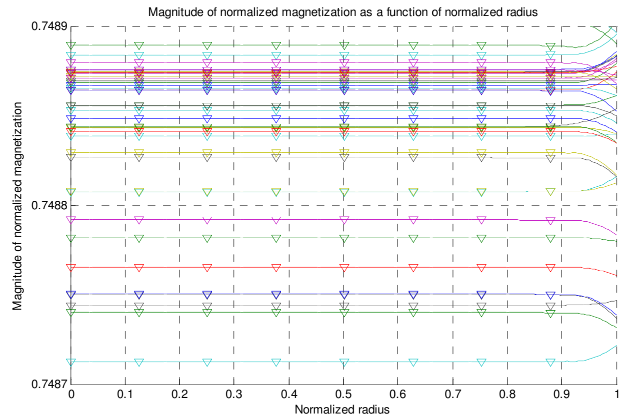

Figure 7. Plot of normalized magnitude of magnetization as a function of normalized radius showing that the magnetization at the wall boundary deviates from the magnetization in the bulk since there is the greatest change in velocity and spin-velocity near the wall. The non-uniformity of the magnetization is very small since the magnitude at the wall changes by ≈1x10^{-4} compared to the magnetization in the bulk of the fluid (0.748).

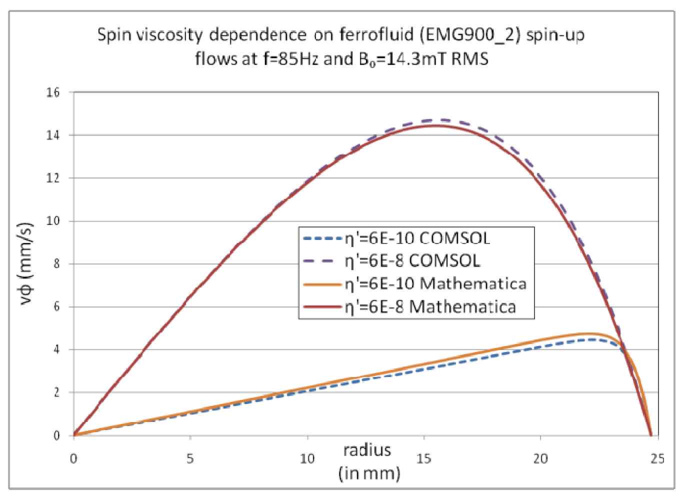

Figure 8. Spin viscosity dependence on velocity flow profiles for the ferrofluid EMG900_2 investigated in Mathematica and COMSOL.

Figure 7 gives the magnetization of the ferrofluid which is slightly non-linear only at the boundary. This implies that the magnetic relaxation equation in Eq. 3, has a small effect on the ferrofluid's magnetization and could be modeled as a linear material. A simple verification can be given by the following calculation

$$\mathbf{H}_{\text{fluid}} = \mathbf{H}_{\text{applied}} - \frac{1}{2}\mathbf{M} \rightarrow \mathbf{M} = \boldsymbol{\chi}\mathbf{H} \rightarrow \mathbf{H}_{\text{fluid}} = \frac{\mathbf{H}_{\text{applied}}}{1 + \frac{1}{2}\chi} \quad \textbf{(27)}$$

If the normalized applied $\mathbf{H}$ field is 1 and χ=1.19, this results in an $|\mathbf{H}_{\text{fluid}}| = 0.627$ and $|\mathbf{M}| = 0.746$. Both these values are obtained in COMSOL as seen in Figure 6 and Figure 7. This subtlety would be difficult to realize and has to be accounted for, as demonstrated in this analysis, when using the one domain scalar potential boundary condition.

In addition, theoretical sweeps of η' can also be done using COMSOL to investigate spin viscosity dependence on velocity flow profiles as seen in Figure 8.

8. Conclusions

This work describes the modeling of spin diffusion dominated ferrofluid flows neglecting demagnetizing effects associated with the shape of the finite height ferrofluid cylinder. It benchmarks experimental results of spin diffusion dominated ferrofluid spin-up flows, with COMSOL simulations and analytical solutions computed using Mathematica.

The experimentally fit values of spin viscosity do result in COMSOL simulations that are in good agreement with experimental results but these values of spin viscosity are many orders of magnitude greater than theoretically derived values [18, 19, 21, 22, 26, 27]. In addition, COMSOL could also be used to investigate the effect of spin-viscosity on ferrofluid flow profiles.

Two different implementations for modeling the magnetic field did give a deeper understanding of the physics of the magnetic relaxation equation and also of the subtlety involved in modeling it as a one domain problem.

9. Acknowledgements

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