Bloch Waves in an Infinite Periodically Perforated Sheet

Waldemar Maysenhölder
Department of Acoustics, Fraunhofer Institute for Building Physics
Nobelstraße 12, 70569 Stuttgart, Germany. waldemar.maysenhoelder@ibp.fraunhofer.de

Abstract: Bloch waves in infinite periodic structures can be conveniently studied by COMSOL. This is demonstrated by a simple, yet rich two-dimensional example: a perforated sheet with square symmetry. The frequencies of Bloch waves are obtained from the solutions of eigenvalue problems with prescribed wave vector. Varying the wave vector results in dispersion relations which form the band structure. The characteristic features of such (propagating or standing) Bloch waves can be examined with the COMSOL plotting and animation tools. In addition, evaluation and visualization of kinetic and potential energy densities as well as of the structure-borne sound intensity vector help to analyze energy aspects of wave propagation. At low frequencies analytical solutions are available for comparison; homogenization leads to an anisotropic effective medium. Thanks to two theorems related to the energy of elastic Bloch waves, the numerical accuracy of some averages over the unit cell may be checked.

Keywords: Perforated sheet, Bloch waves, band structure, structure-borne energy density, structure-borne intensity.

1. Introduction

Almost thirty years ago the author began to theoretically investigate periodic components in building acoustics, e.g. masonry walls. The general consideration of energy aspects of structure-borne sound eventually resulted in the monograph [1]. (It is written in German; unfortunately, there is no English translation, however, a French online version [2] is available.) The chapter on periodic structures contains general formulations including formulas for energy densities and intensities of Bloch waves. A low-frequency approximation, which leads to an analytic solution for arbitrary one-dimensional periodic structures, is exemplified with the two-dimensional structure shown in Fig. 1. Despite its simple appearance the richness of the results is substantial and quite instructive.

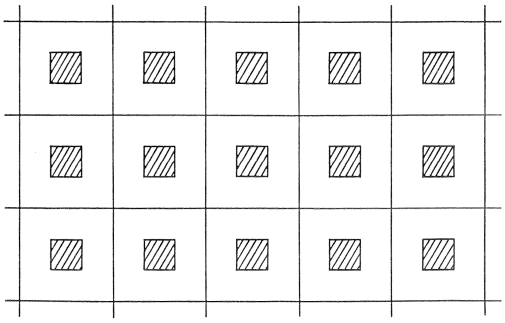

Figure 1. 2D periodic structure "Perforated Sheet".

Meanwhile, COMSOL has become a convenient tool for handling infinite periodic structures. After definition of the unit cell of the periodic structure the COMSOL model is completed with few mouse clicks. Of course, the scope of results which can be calculated and visualized by COMSOL is much larger than shown in [1], since the COMSOL model is not restricted to low frequencies.

However, generating results and understanding them are different matters. Knowledge of the underlying theory of Bloch waves in elastic structures greatly helps with interpretation. The following section 2 recollects some of the mentioned "old" results, which are then contrasted with COMSOL results in section 3. The combination of numerical tools with a profound theoretical knowledge favors correct conclusions and successful applications (section 4).

2. Review of Analytical Results

2.1 Two Theorems

Two theorems related to the energy density and the intensity of Bloch waves [1, 3] can be useful for the analysis of energy and energy transport aspects.

(i) **Rayleigh's principle for progressive waves** (see e.g. [4]) states that under certain conditions the averages of kinetic and potential energy densities of progressive waves are equal. For Bloch waves the energy densities e_{kin} and e_{pot} have to be averaged both over time t (over a period T) and over space $\vec{r}$ (over a unit cell uc) in order to arrive at the equality

$$\left\langle\left\langle e_{\text{kin}}(\vec{r},t)\right\rangle_T\right\rangle_{\text{uc}} = \left\langle\left\langle e_{\text{pot}}(\vec{r},t)\right\rangle_T\right\rangle_{\text{uc}}. \quad (1)$$

(ii) **The group velocity is equal to the energy velocity deduced from averages over the unit cell.** The group velocity $\vec{C}$ is equal to the gradient of the (circular) frequency ω with respect to the wave vector $\vec{k}$,

$$\vec{C} \equiv \nabla_{\vec{k}}\, \omega\,, \quad (2)$$

while the energy velocity $\vec{c}_e$ is defined as the ratio of the intensity (which is the time average of the energy flux density) and the time average of the total energy density. For a Bloch wave with Bloch wave vector $\vec{k}$ the two velocities are equal, if the energy velocity is defined with averages over the unit cell:

$$\nabla_{\vec{k}}\, \omega \equiv \vec{C} = \vec{c}_{\text{e\_uc}} \equiv \frac{\left\langle \vec{I}(\vec{r})\right\rangle_{\text{uc}}}{\left\langle\left\langle e_{\text{kin}} + e_{\text{pot}}\right\rangle_T\right\rangle_{\text{uc}}}. \quad (3)$$

NB: The phase velocity in the direction of $\vec{k}$ is obtained by

$$c = \frac{\omega}{\left|\vec{k}\right|}. \quad (4)$$

2.2 Low-Frequency Approximation

With the assumption that at low frequencies the phase velocity c does not depend on the frequency (but still depends on Bloch-wave type and usually also on propagation direction), an analytic approximation was worked out [1], which leads to an infinite system of linear equations. Explicit expressions for the averages in (3) were also obtained. With the aid of Rayleigh's principle for Bloch waves (1) an analytical solution was derived for general one-dimensional periodic structures. Numerical evaluations for the two-dimensional structure of Fig. 1 included deformations, energy densities and intensities for some propagation directions. Figs. 2 to 5 show graphs from the short report [5] published before the book [1].

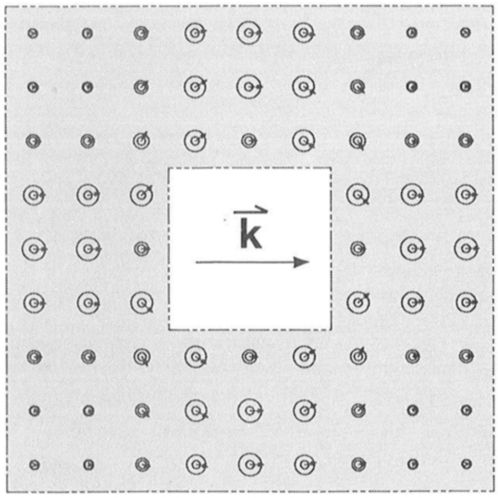

Figure 2. Intensities (arrows) and time-averaged energy densities (circles) for the mainly transversally polarized Bloch wave with wave vector $\vec{k}$ at low frequencies. Inner circles: kinetic, outer circles: total energy density. The radii of the circles are proportional to the energy density.

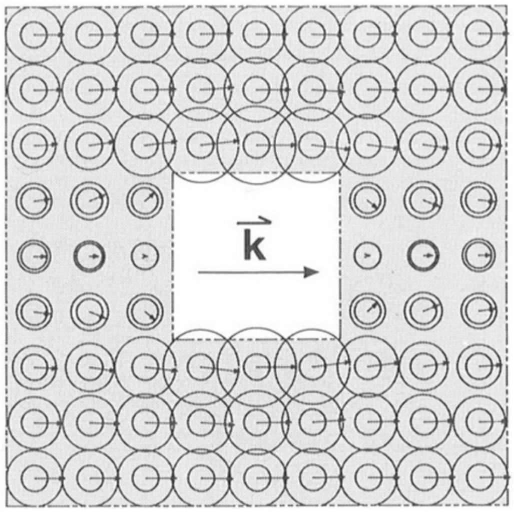

Figure 3. As Fig. 2, but for the mainly longitudinally polarized Bloch wave.

Looking at the circles in Fig. 3 it is clear that the (time averages of the) kinetic and potential energy densities can be quite different locally: Left and right from the hole there is little potential energy. In order to satisfy Rayleigh's principle (1), this is compensated by an excess of potential energy above and below the hole.

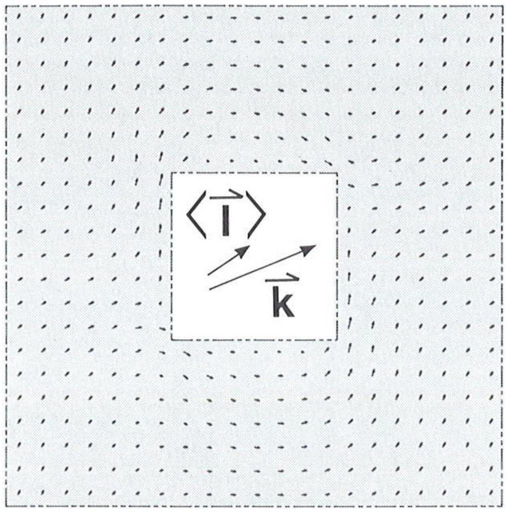

Figure 4. Intensities (arrows) for the mainly transversally polarized Bloch wave with wave vector $\vec{k}$ at 22.5° relative to the horizontal unit-cell edge at low frequencies. In the hole the direction of the unit-cell-averaged intensity is shown.

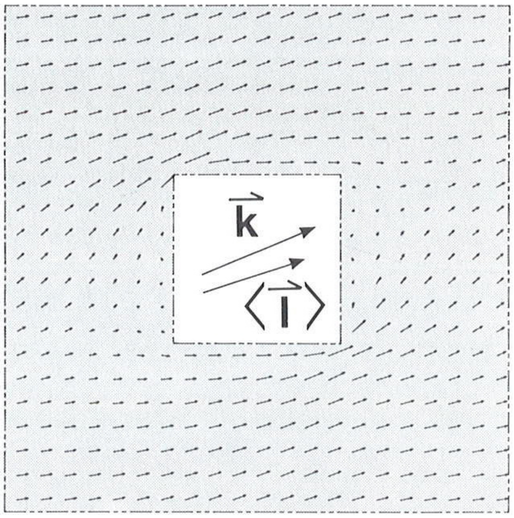

Figure 5. As Fig. 4, but for the mainly longitudinally polarized Bloch wave.

Figs. 4 and 5 show the energy flow in more detail and moreover illustrate that in case of propagation in non-symmetry directions the intensity averaged over a unit-cell is not parallel to the propagation direction, a characteristic feature of homogeneous, but anisotropic media.

2.3 Homogenization

At low frequencies the Bloch waves resemble plane waves. The periodic medium "behaves" like a homogeneous medium, however with direction-dependent properties. The "effective" moduli of such an "equivalent" anisotropic medium can be obtained from the (frequency-independent) phase velocities (4) along several propagation directions. In the present example with square symmetry the determination of the three effective moduli is rather simple. The general relation between phase velocities and effective moduli (up to 21) was given by Norris [6].

Once the effective moduli are known, the periodic medium can – as for the calculations – be replaced by the equivalent homogeneous anisotropic medium, which is much simpler to deal with: Computation of polarization, slowness surface, group velocity, energy densities and intensity are comparatively easy.

3. COMSOL Results

3.1 Model Setup

How COMSOL can be used for infinite periodic structures was communicated recently by Elabbasi [7]. The essential feature is "Floquet Periodicity", which is found among the boundary conditions ("Connections" >> "Periodic Condition" >> "Type of periodicity"). In the present example (Fig. 6) two such "Connections" are required, one for the x-direction and one for the y-direction. Visualization of results over more than one unit cell is achieved by the "Data Set" "Array 2D". Here one should not miss to check the box "Floquet periodicity" under "Advanced" and enter the correct wave vector components. (Caution! The coordinates "X:" and "Y:" are plot coordinates, which are not necessarily identical to the global model coordinates. If – e.g. in case of 2D periodicity and a 3D unit cell – an "Array 2D" is generated in a "Cut Plane", it may happen that after "X:" the wave vector component k_y has to be entered!)

As to the term "Floquet periodicity": The displacement field of a propagating Bloch wave is not periodic in the strict mathematical sense of the word. It therefore appears unfortunate to call a non-periodic quantity periodic. "Floquet-Bloch Condition" would be more appropriate.

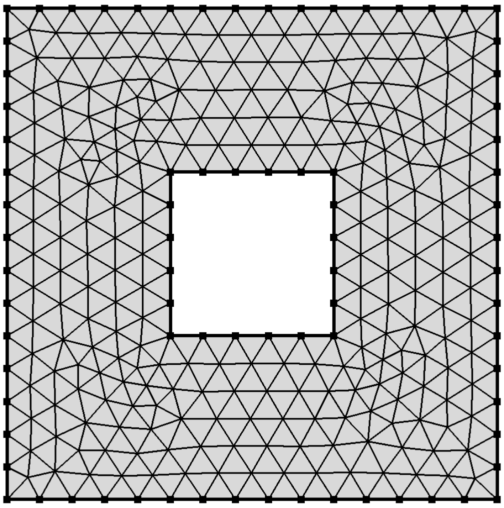

Figure 6. Unit cell of the 2D periodic structure with "physics-controlled, normal" mesh. Outer edge: 3 cm; inner edge: 1 cm. Material properties: Young's modulus: 10 MPa; Poisson's ratio: 1/6; mass density: 1500 kg/m³.

With reference to masonry walls mentioned in the introduction the current COMSOL version 5.2 does not support the modeling of infinite periodic structures without rectangular (orthotropic) symmetry. Brickwork with "oblique" periodicity like in Fig. 7 is excluded, unless the offset ("lap") between rows ("courses") of bricks is one half of the longer side of a brick and nothing else breaks the rectangular symmetry. In that case, however, the smallest unit cell is two bricks large. (Similarly, a hexagonal structure with rectangular symmetry can be modeled by a rectangular lattice with rectangular unit cell, the size of which comprises two basic hexagons.)

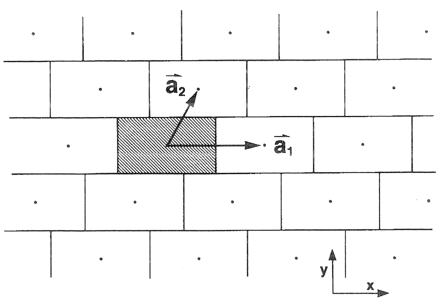

Figure 7. Non-orthogonal lattice corresponding to the masonry type "raking stretcher bond" (in German: "Schleppender Läuferverband").

The "Study" type needed for the determination of Bloch waves is "Eigenfrequency". This term is rather reserved for vibrations of finite structures. In the infinite case with waves, however, one is faced as well with – infinitely many – eigenvalue problems. The Bloch waves are so to speak the modes of infinite periodic structures. Yet in contrast to eigenvibrations, a wave possesses an identifier additional to frequency: the wave vector. As a consequence one has to specify the wave vector, i.e. propagation direction and wavelength, for the definition of the eigenvalue problem.

In order to obtain an overview of the properties of the Bloch-wave "families" of a particular periodic structure, the wave-vector space is sampled along a few selected directions by means of a "Parametric Sweep". If desired, the sweeping parameter (called "p" in the present example) can be cleverly defined as to cover several directions in one run [7].

3.2 Band Structure

The band structure contains information on Bloch-wave dispersion, i.e. the dependence of the frequency on the wave vector. Fig. 8 shows results for the x-direction up to p = 0.5, which means up to

$$k_x = \frac{\pi}{L}, \tag{5}$$

where the first Brillouin zone ends (*L*: edge length of unit cell). Since the band structure is periodic in wave-vector space, it is sufficient to investigate the first Brillouin zone [8].

The two lines starting at p = 0 and zero frequency are called "acoustic branches". They correspond to transversal and longitudinal plane waves in the homogeneous sheet (i.e. without holes). All other – infinitely many – branches are called "optical" [8]. There is a "band gap" (or "stop band") around 1.4 kHz: In this frequency region Bloch waves propagating along the x-direction do not exist.

At about p = 0.42 it looks as if the green branch and the red one "want" to cross. This is in fact the case, but since COMSOL does not care about branch crossing, the line and color assignment fails.

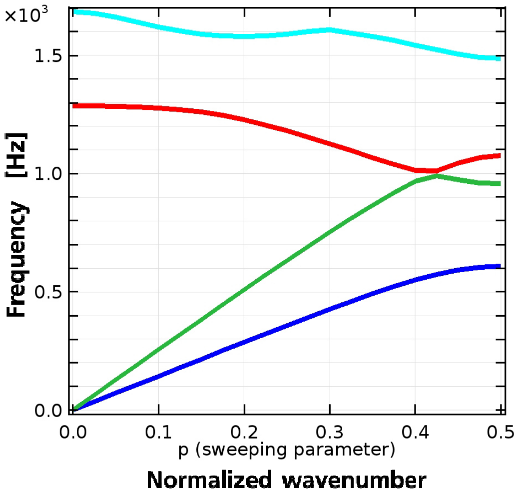

Figure 8. Band structure for wave vectors along the x-direction.

With Eq. (4) the phase velocity of a Bloch wave can be easily extracted from the band structure, while determination of the group velocity via the gradient according to (3) demands more effort. However, for symmetry directions, here e.g. the x-direction, this task simplifies to quantify the slope of the dispersion curve. Negative slopes in the red and blue branch lines in Fig. 8 imply negative group velocities, i.e. phase and energy travel in opposite directions.

3.3 Bloch-Wave Visualization

COMSOL provides convenient and powerful tools for visualization and analysis of Bloch waves. Animations are particularly helpful for detailed studies of motions and deformations. Figs. 9 to 11 show snapshots of Bloch waves (color encodes “solid.disp”).

Fig. 9 clearly demonstrates that the displacement associated with a Bloch wave is not sinusoidal. Bloch waves can be considered as plane waves modulated by a perturbation with the periodicity of the structure. For the acoustic branches this deviation from a plane wave becomes weaker with decreasing wavenumber.

At the edge of the first Brillouin zone at p = 0.5 the slopes of the branches in Fig. 8 become zero. This implies zero group velocity and thus standing waves (Fig. 11). Here the periods of the unperturbed plane wave and the structure become commensurable and the Bloch wave is periodic.

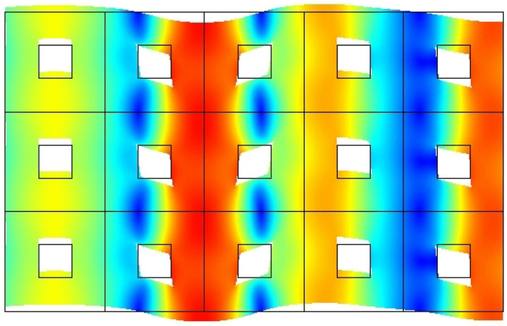

Figure 9. Bloch wave of shear type (blue branch at p = 0.35 and 491 Hz).

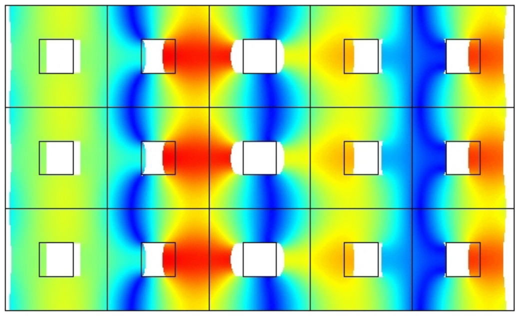

Figure 10. Bloch wave of longitudinal type (green branch at p = 0.35 and 866 Hz).

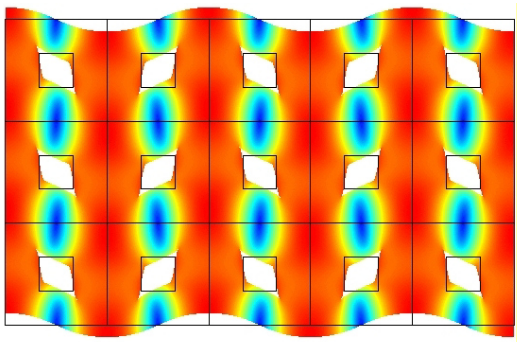

Figure 11. Standing Bloch wave of shear type (blue branch at p = 0.5 and 609 Hz).

3.4 Energy Densities and Intensity

Acoustic phenomena are not completely understood if energy aspects are ignored. This is particularly true for structure-borne sound in inhomogeneous media. Therefore it is worthwhile to look at energy densities and energy transport.

Figs. 12 to 15 illustrate such quantities for the light-blue optical Bloch wave at $p = 0.1$ and 1619 Hz. Whereas the kinetic energy density distribution is rather smooth at small scales, the potential counterpart (strain energy) is highly concentrated at the corners of the hole.

Fig. 15 shows with streamlines and arrows how the energy flows around the hole. Two features are conspicuous: (i) there are intensity vortices and (ii) the net energy flow is from the right to the left, i.e. opposite to the wave-vector direction. Comparisons with graphs based on the low-frequency approximation turned out satisfactory (see Figs. 5 and 16 as example).

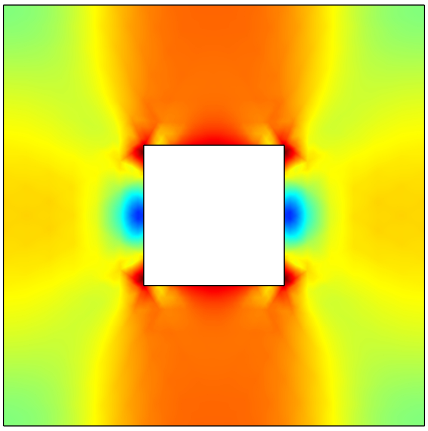

Figure 14. As Fig. 12, but for total energy.

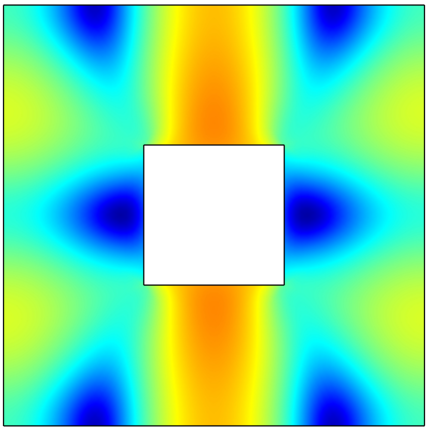

Figure 12. Time average of kinetic energy density (light-blue branch at $p = 0.1$ and 1619 Hz).

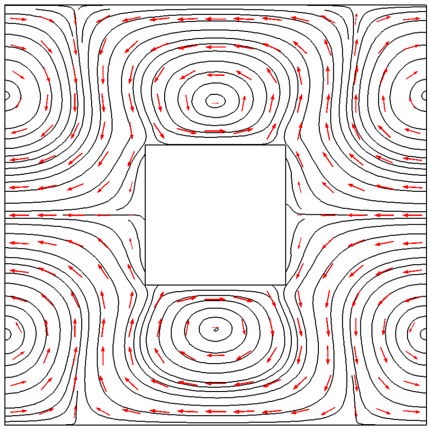

Figure 15. Structure-borne sound intensity (light-blue branch at $p = 0.1$ and 1619 Hz).

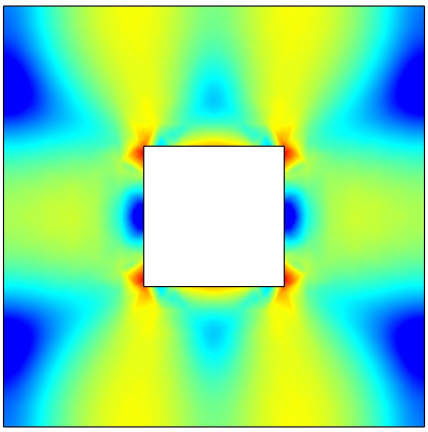

Figure 13. As Fig. 12, but for potential energy.

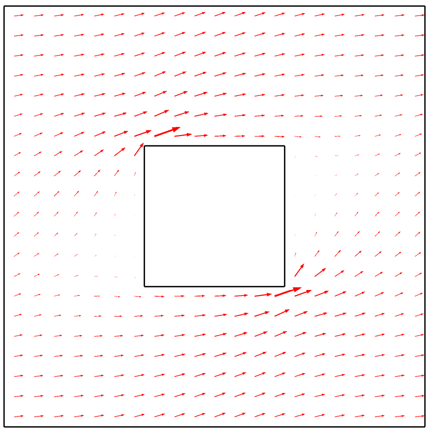

Figure 16. Structure-borne sound intensity for oblique propagation along 22.5° (for comparison with Fig. 5).

4. Applications

Metamaterials [9-10] with periodic microstructures are currently much in vogue. Thus, there are many opportunities for Bloch-wave studies with COMSOL. Although real structures are always finite, the infinite theoretical model is of great help for analysis and optimization. Since only one unit cell has to be dealt with, the numerical effort is relatively small. If homogenization is appropriate, it can drastically simplify the model of the finite structure. Other applications may concern quite practical objects like sports-hall floors with periodically distributed resilient supports.

Further, the Bloch-wave concept can also be applied to the sound transmission problem of an infinite 2D-periodic partition between two homogeneous fluid half-spaces, since parallel to the partition the half-spaces trivially possess the same periodicity. With the additional items “Background Pressure Field” and “Perfectly Matched Layer” for the infinite extension perpendicular to the partition a COMSOL model for this transmission problem is readily obtained.

5. Conclusion

COMSOL is a valuable and powerful tool for investigating periodic structures in a wide range of frequencies. Former analytic low-frequency approximations are easily reproduced numerically. Detailed consideration of the energy aspect might open new perspectives, also and especially for metamaterial research. Two pertinent theorems are helpful both for understanding acoustic phenomena and for simplifying and checking numerical calculations.

6. Acknowledgements

This work was performed within the project “Acoustic Metamaterials” of the Fraunhofer program International Cooperation and Networking (ICON). Besides, the author is grateful to the COMSOL support team for kind and valuable assistance.

7. References

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