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A Mixed Boundary Value Problem That Arises in the Study of Adhesively Bonded Structures

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Given a physics-based problem, we often have choices in deciding

a) which Reduced-Order Model to use

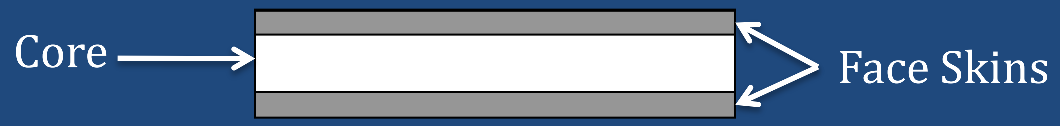
b) which mathematical method to use to analyze the ROM

Examples come from Ocean Modeling, Electromagnetism, and Structural Mechanics

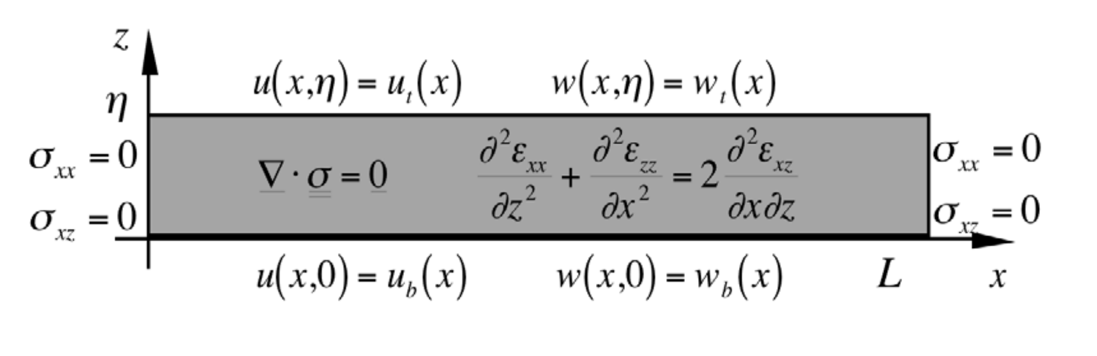
Typical Problem Genesis:

- Adhesively Bonded Joints

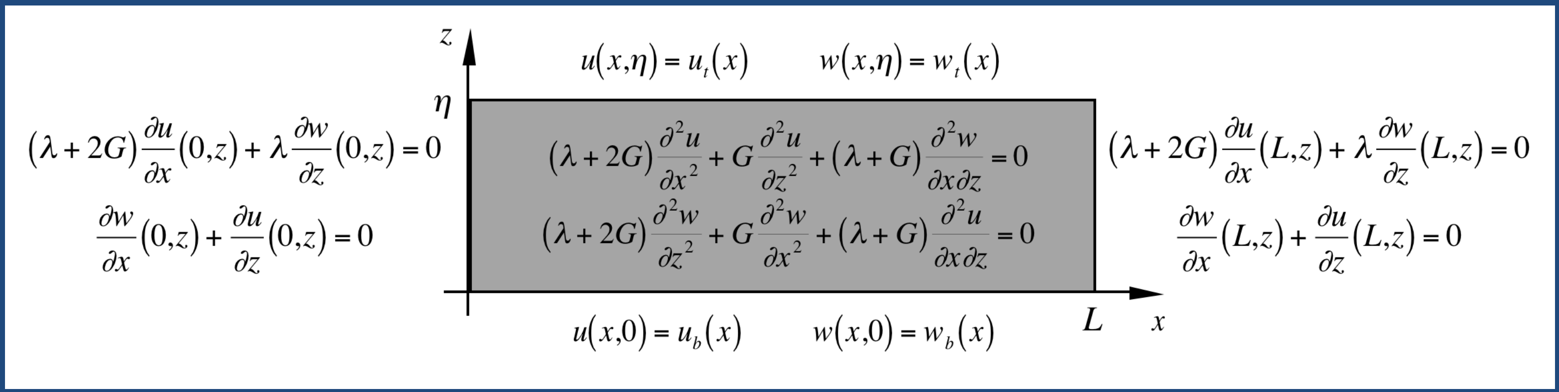
- Aerospace Sandwich Structures



Boundary Value Problem: Mixed form in terms of Stress, Infinitesimal Strain, and Displacement



Boundary Value Problem: Expressed in terms of Displacement

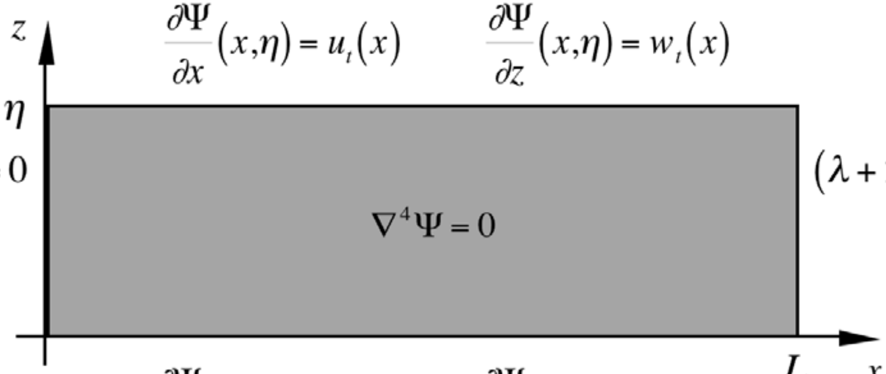


Boundary Value Problem: Expressed in terms of Displacement Potential

$$\frac{\partial \Psi}{\partial x}(x,\eta) = u_t(x) \qquad \frac{\partial \Psi}{\partial z}(x,\eta) = w_t(x)$$

$$(\lambda + 2G)\frac{\partial^2 \Psi}{\partial x^2}(0,z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(0,z) = 0$$

$$\frac{\partial^2 \Psi}{\partial x \partial z}(0,z) = 0$$



$$(\lambda + 2G)\frac{\partial^2 \Psi}{\partial x^2}(L,z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(L,z) = 0$$

$$\frac{\partial^2 \Psi}{\partial x \partial z}(L,z) = 0$$

$$\frac{\partial \Psi}{\partial x}(x,0) = u_b(x) \qquad \frac{\partial \Psi}{\partial z}(x,0) = w_b(x)$$

$$\frac{\partial \Psi}{\partial x}(x,z) = u(x,z)$$

$$\frac{\partial \Psi}{\partial z}(x,z) = w(x,z)$$

Boundary Value Problem: Expressed in terms of Airy's Stress Function and Solution Ansatz

$$\nabla^4\Phi = 0;\{0 \le x \le L, 0 \le z \le \eta\}$$

$$\frac{\partial^2\Phi}{\partial z^2}(0,z) = 0 \quad \frac{\partial^2\Phi}{\partial x\partial z}(0,z) = 0$$

$$\frac{\partial^2\Phi}{\partial z^2}(L,z) = 0 \quad \frac{\partial^2\Phi}{\partial x\partial z}(L,z) = 0$$

$$\left[\frac{1}{E}\int\left(\frac{\partial^2\Phi}{\partial x^2} - \nu\frac{\partial^2\Phi}{\partial z^2}\right)dz\right]\Bigg|_{z=0} + C_1(x) = w_b(x)$$

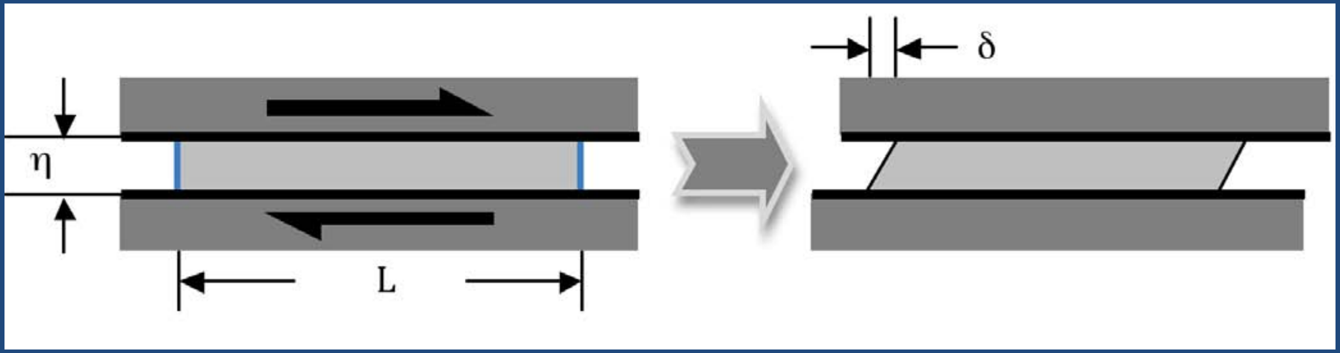
$$\left[\frac{1}{E}\int\left(\frac{\partial^2\Phi}{\partial x^2} - \nu\frac{\partial^2\Phi}{\partial z^2}\right)dz\right]\Bigg|_{z=\eta} + C_1(x) = w_t(x)$$

$$\left[-\frac{1}{G}\int\left(\frac{\partial^2\Phi}{\partial x\partial z}\right)dz - \frac{1}{E}\int\int\left(\frac{\partial^3\Phi}{\partial x^3} - \nu\frac{\partial^3\Phi}{\partial x\partial z^2}\right)dzdz\right]\Bigg|_{z=0} + C_2(x) = u_b(x)$$

$$\left[-\frac{1}{G}\int\left(\frac{\partial^2\Phi}{\partial x\partial z}\right)dz - \frac{1}{E}\int\int\left(\frac{\partial^3\Phi}{\partial x^3} - \nu\frac{\partial^3\Phi}{\partial x\partial z^2}\right)dzdz\right]\Bigg|_{z=\eta} + \eta\frac{dC_1}{dx}(x) + C_2(x) = u_t(x)$$

$$\Phi(x,z) = A_0(x) + zA_1(x) + \sum_{n=1}^{\infty}F_n(x)\sin\left(\frac{n\pi z}{\eta}\right)$$

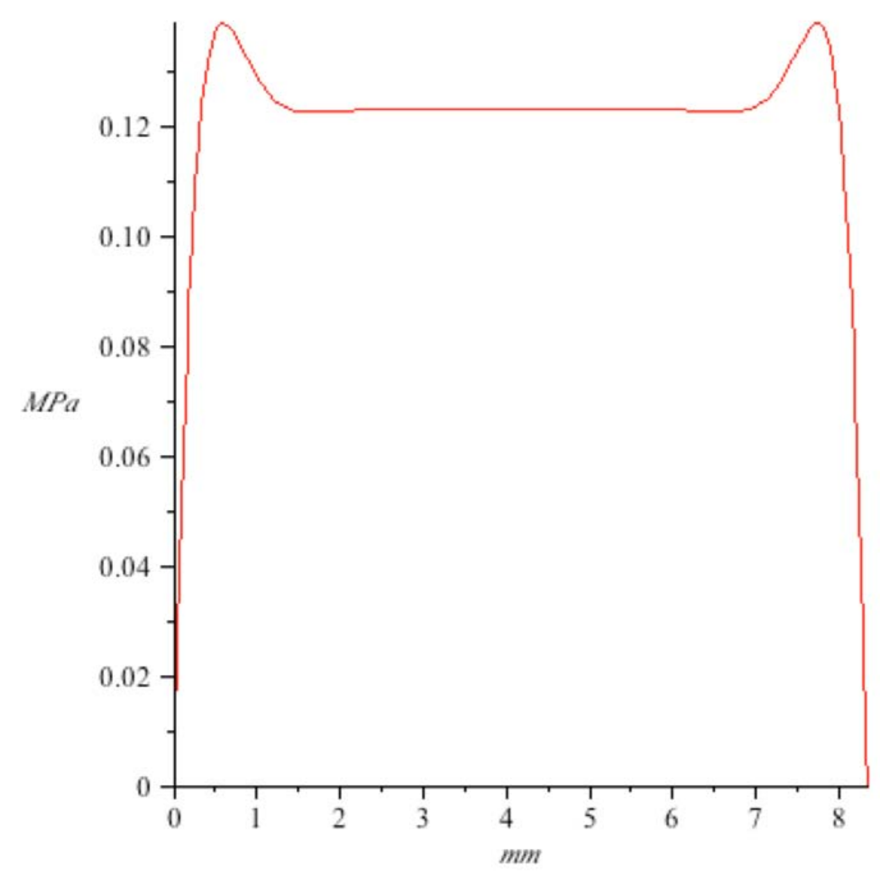
Example Loading Case: Displacements, Geometry, and Properties



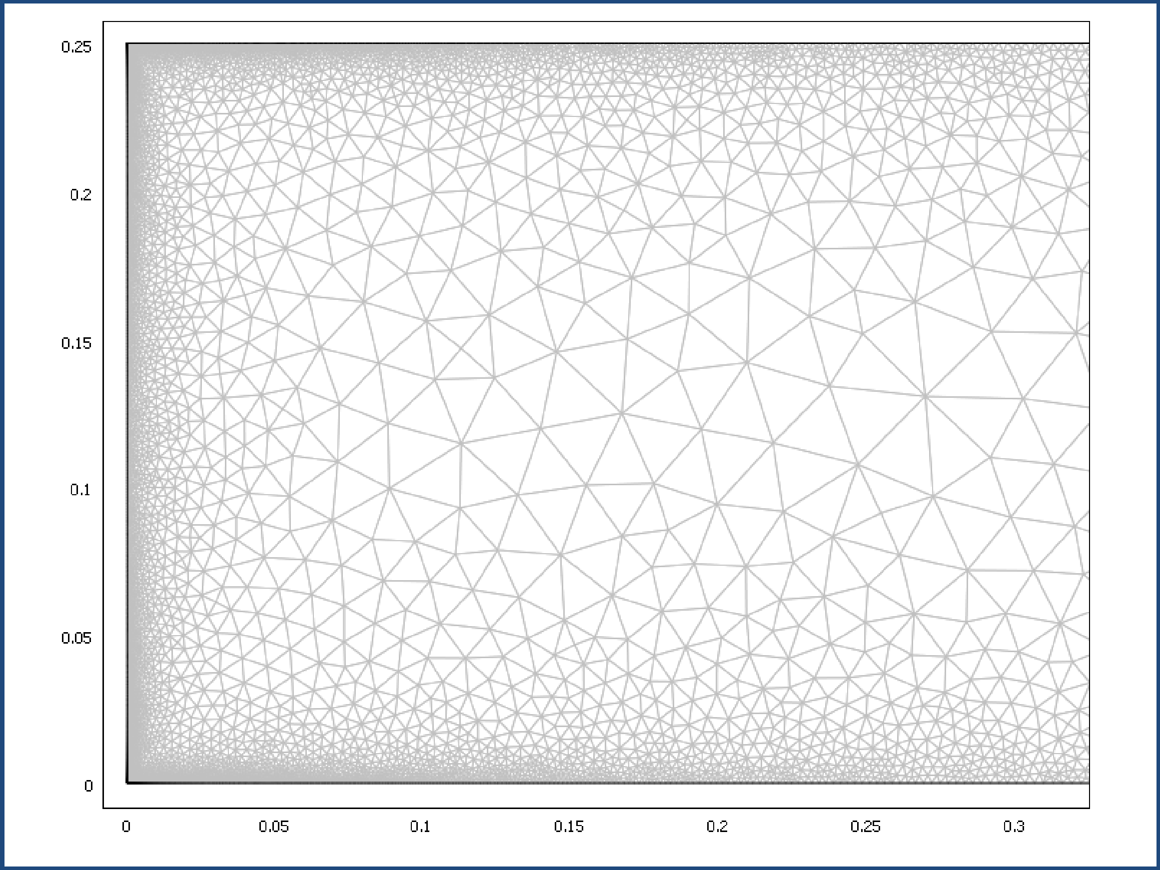
E	υ	L	η	δ
344.6 MPa	0.3	8.333 mm	0.25 mm	0.00025 mm

$u_b(x)$	$u_t(x)$	$w_b(x)$	$w_t(x)$
0 mm	0.00025 mm	0 mm	0 mm

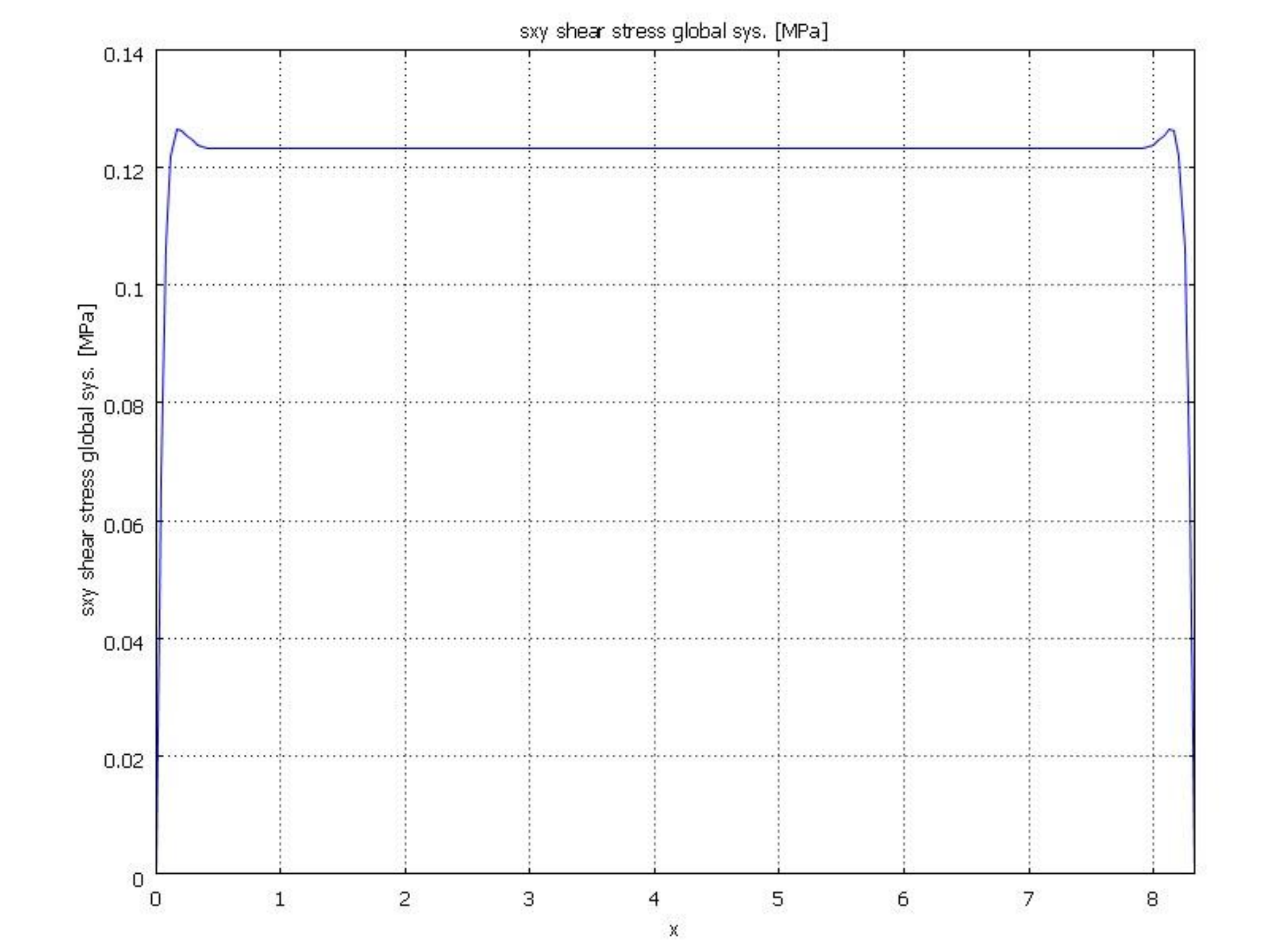
Spectral-Collocation Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x,\eta/2)$ and Interfacial Shear Stresses $\sigma_{xz}(x,0)$



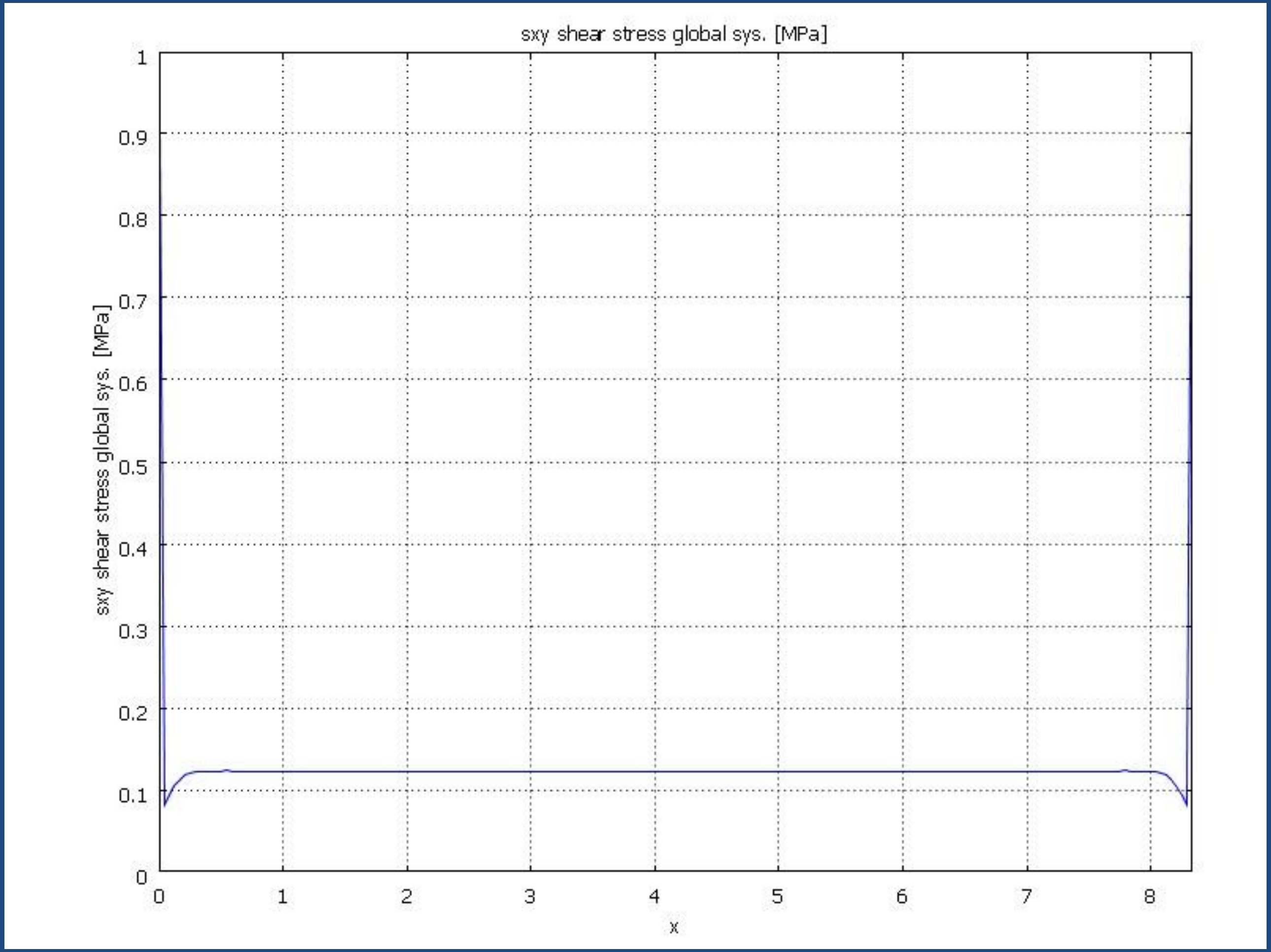
COMSOL Structural Mechanics Mesh: Shown in vicinity of stress free surface



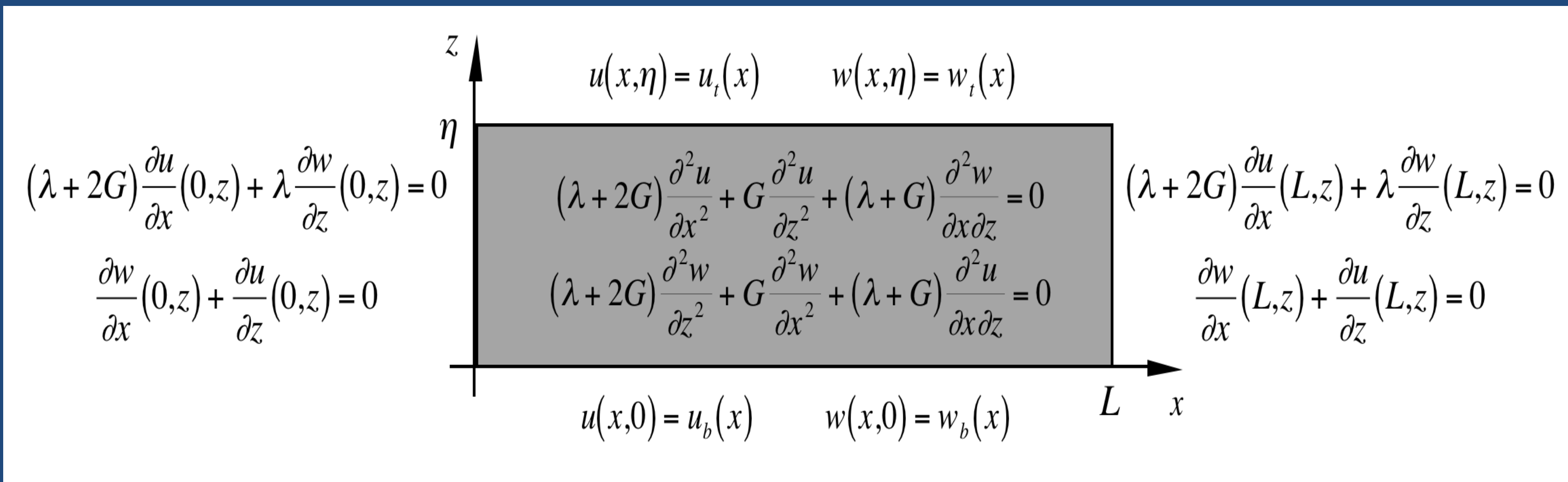
COMSOL Structural Mechanics 2D Plane-Stress Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x,\eta/2)$



COMSOL Structural Mechanics 2D Plane-Stress Analysis Results: Interfacial Shear Stress $\sigma_{xz}(x,0)$



Back to Displacement BVP

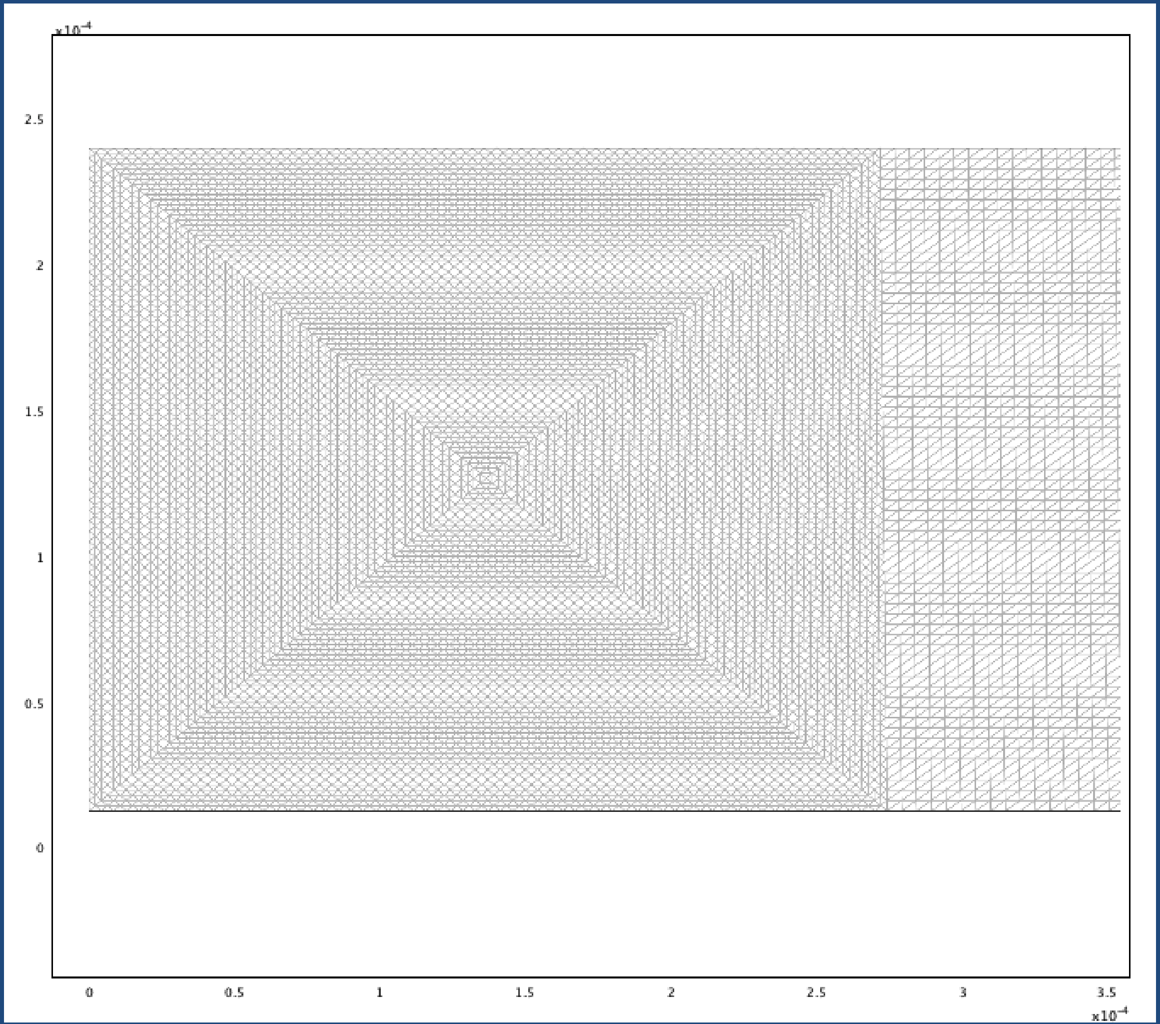


$$\nabla \cdot \Gamma = \mathbf{F}$$

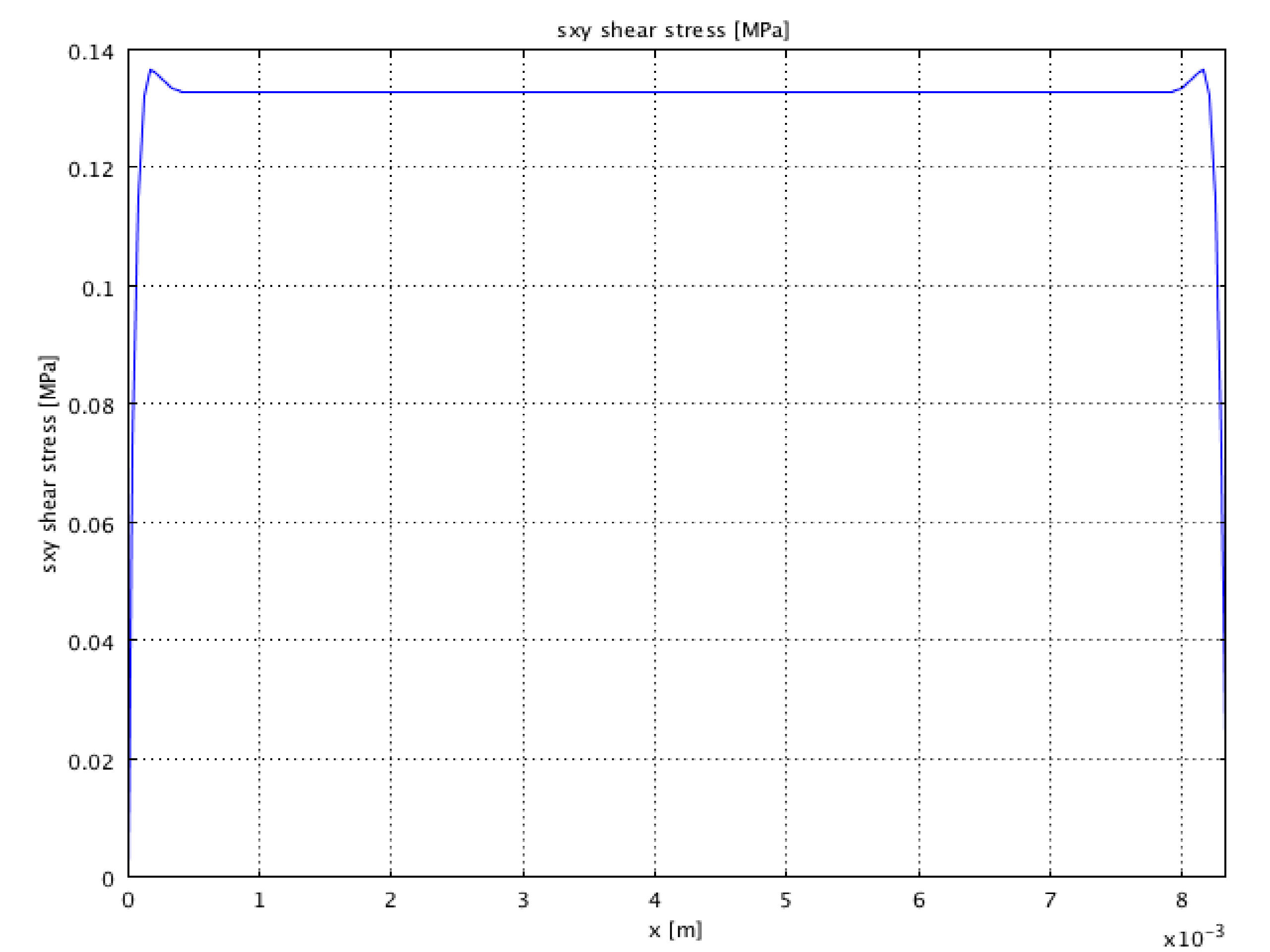
Use COMSOL's General PDE Solver

Gamma is a 2 x 2 tensor

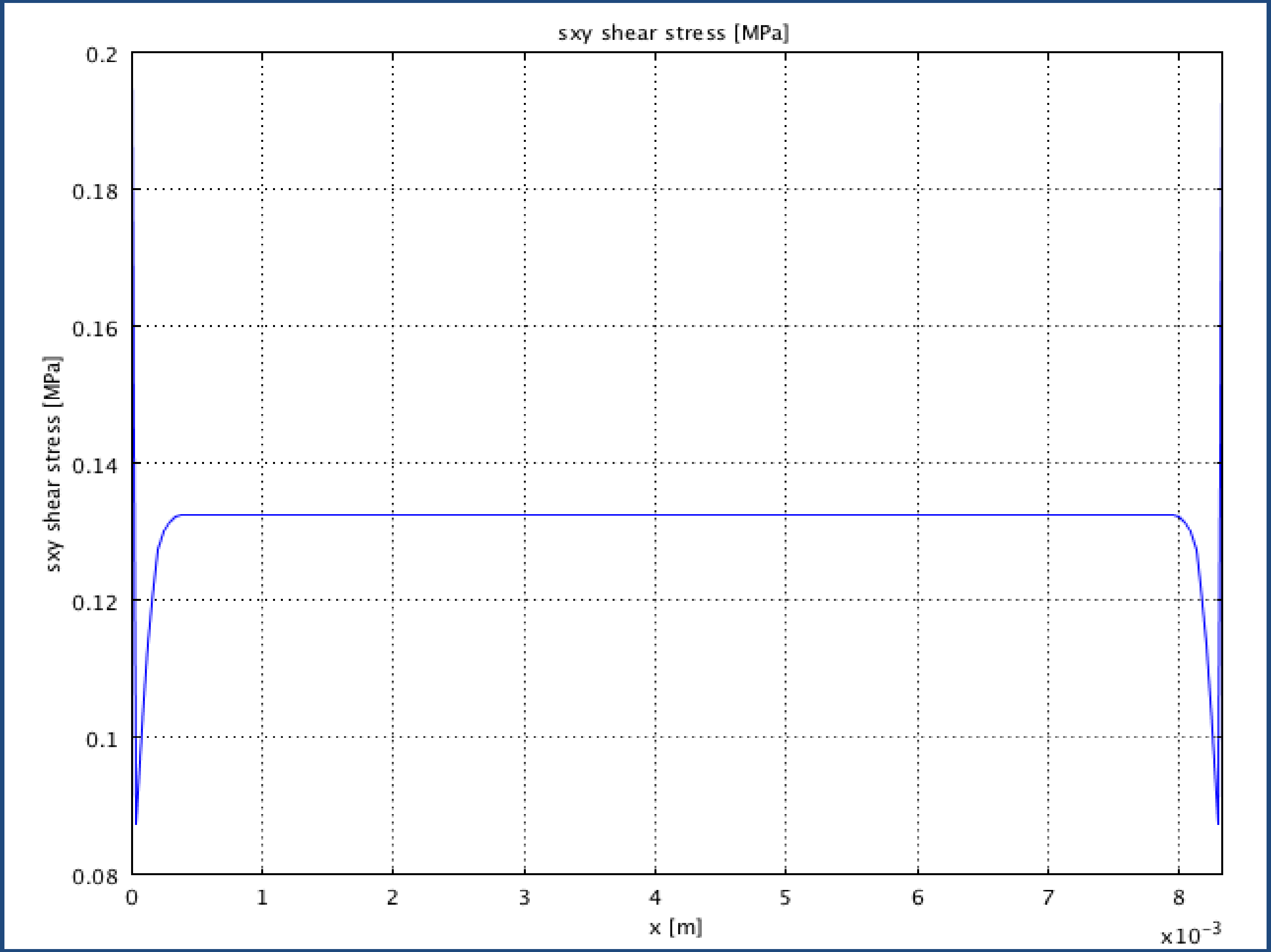
COMSOL PDE (General Form) Solver Mesh: Shown in vicinity of stress free surface



COMSOL PDE Solver Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x,\eta/2)$



COMSOL PDE Solver Analysis Results: Interfacial Shear Stress $\sigma_{xz}(x,0)$



Some Conclusions/Observations

The Spectral Collocation method describes a shear stress that does not have a singularity at the corners; this seems to be the expected result from a mechanical point of view.

The Structural Mechanics result seems to suffer from artificially large singularities at the corners; did we implement it poorly?

The General PDE solver seems to be a natural way to pose this pose this problem In COMSOL. It gives good result, although it seems to still suffer from some level of Numerical singularity at the corners.