Analytical Solution for the Steady Poroelastic State under Influence of Gravity

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Introduction

Poroelastics combines the fields of hydraulics and mechanics. Hydraulics and mechanics are linked in a two-way coupling, also referred to as HM coupling. Major application fields are ground stability, foundation of buildings, waste injection, CO_2 injection and steam assisted gravity drainage. In many of these applications gravity has to be taken into account.

Studies of poroelastic systems increasingly utilize numerical modelling. Especially for geotechnical applications models are set up as a tool to understand phenomena in porous media that deform due to changes of the hydraulic regime, or in which the flow field is affected by changes of the stress regime. The undisturbed state, which is often used as initial condition or for reference, is hydrostatic i.e. shows linear pressure increase with depth. Zero is the constant value of deformation, the mechanical state variable, that corresponds with hydrostatics.

If gravity as outer force is considered, the zero deformation does not match with the hydrostatic state anymore. For modelling that means that some efforts have to be made to find a steady state solution. Mostly some pre-runs of the numerical model are performed to obtain the solution. Here it is shown that an analytical solution provides an alternative approach to numerical pre-runs.

Starting from the fundamental descriptions of poroelastic systems with HM coupling the analytical solution for the geostatic state under influence of gravity is derived. Vertical deformation, changing with depth following a quadratic regime, matches with the hydrostatic state.

The derived solution is valid for a homogeneous poroelastic system with constant parameters. Using COMSOL Multiphysics it is demonstrated that numerical and analytical solutions coincide. We use the structural mechanics module of the COMSOL product suite. Simulations of the transient development show solutions converging to the analytical solution.

The solution delivers a formula for the vertical deformation, and thus for the shrinking of a hypothetical system that was originally not exposed to gravity under the influence of the gravity force. From this we develop the solution for the inverse problem: what is the thickness of a system that is not exposed to an outer force, when the thickness of the system under gravity regime is given.

The so derived solution can be beneficial for numerical modelling of poroelastic systems under the influence of gravity. The analytical solution can be used as initial condition. Thus simulations of transient developments do not require an artificial initialization phase to obtain an approximation of the steady state. Also a pre-run for direct solution of the steady state, as alternative to transient initialization, becomes obsolete.

Moreover the analytical steady state solution can be used as a reference to compare transient deformations during the simulation. The presented analytical solution simplifies numerical simulations of poroelastic systems. An application case was already presented by Holzbecher (2014).

Governing Equations and Solutions

The 1-dimensional description of the geostatic state, using linear constitutive stress-strain relationship, is given by

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\cdot\frac{\partial^2 w}{\partial z^2}=\alpha\frac{\partial p}{\partial z}+\rho g \qquad (1)$$

with parameters Young modulus E, Poisson ratio ν, the Biot or Biot-Willis coefficient α, density of the fluid/solid porous system ρ and acceleration due to gravity g. The dependent variable w is the deformation in vertical direction. z is the spatial variable in vertical direction, opposite to the direction of gravity. In the coupled poroelastic system the variable of pore pressure p is determined also by the hydraulic regime. The mathematical description of the hydraulic regime is derived from Darcy's law and the principle of mass conservation – for details see for example: Wang (2000) and Ingebritsen *et al.* (2007).

It is assumed that the solution for the mechanical steady state is connected with the hydrostatic solution, given by:

$$p = -\rho_f g z \tag{2}$$

with fluid density ρ_f. Introducing this into equation (1) yields:

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \cdot \frac{\partial^2 w}{\partial z^2} = (-\alpha\rho_f + \rho) g \tag{3}$$

As it is further assumed here that all parameters are constants, the solution w of eq. (3) is a quadratic function of z. As boundary conditions are required:

$$w(0) = 0 \text{ and } \frac{\partial w}{\partial z}(H) = 0 \tag{4}$$

where z=0 denotes the base of the modelled layer and H it's thickness in a system without gravity. With conditions (4) the solution reads:

$$w(z) = \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} (-\alpha\rho_f + \rho) g z (z - 2H) \tag{5}$$

The maximum deformation is given at the top of layer, i.e. the absolute value of $w(H)$:

$$w_{\max} = -\frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} (-\alpha\rho_f + \rho) g H^2 \tag{6}$$

Lets define H_0 as the height of the deformed steady state under the influence of gravity. In contrast H represents the thickness of the layer in a hypothetical environment without gravity. The two heights are connected by the formula:

$$H = H_0 + w_{\max} \tag{7}$$

Combining equations (6) and (7) a quadratic equation for w_{max} is obtained:

$$w_{\max} = -\tfrac{1}{2} A (H_0 + w_{\max})^2 \tag{8}$$

with

$$A = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} (-\alpha\rho_f + \rho) g \tag{9}$$

The solutions of equation (8) are:

$$w_{\max} = \frac{1 - AH_0}{A} \pm \sqrt{\frac{(1 - AH_0)^2}{A^2} - H_0^2} \tag{10}$$

Only the negative sign in front of the square root gives reasonable solutions. This is a rough outline of the derivation presented in more detail by Holzbecher (2017). In that publication Holzbecher also deals with the more complex case of a system of horizontal layers. This contribution treats the single layer situation only and the focus lies on the limitations of the presented approach with respect to the given parameters.

Utilization in Numerical Modeling

In most application studies of poroelastics the considered porous medium reacts to a changed geomechanical or hydraulic regime, the forcing, and approaches another steady state if the changed conditions persist. In such simulations the initial condition is of concern, which is a steady state that has developed within the system without the outer forcing.

If gravity is not taken into account the steady state is given by zero deformation and hydrostatic pressure. This can be used as initial state. But the same simple state is not valid, if gravity is considered. Then the initial state corresponding with hydrostatic pressure is a nonzero vertical deformation. More precisely it is a quadratic function of depth, as shown above. Its exact formula depends on the hydraulic and geomechanical parameters.

Modellers of the transient states as described, may thus follow one of three strategies:

(1) In the simulation they use an initialization period, in which the numerical model approaches the 'natural' steady state. Nopper *et al.* (2012) follow such a strategy using a *transient pre-run*.

(2) They use a pre-run solving for the steady state. The solution of the pre-run is stored and then utilized as initial condition for the transient simulation runs. Examples for such *stationary pre-runs* can be found in the publications of Altmann (2010) and Chamani (2013).

(3) The analytical solution derived above can be used as initial condition.

If applied correctly all three strategies provide the same solution. However, there are advantages and disadvantages concerning the options. The initial time period in (1) does not reflect the physical behaviour of the system, because the initial condition of the numerical run is artificial. The length of the initialization period is not known, but must be long enough to ensure that the steady state is sufficiently good approximated. It may require several runs to determine an appropriate length of the initialization period. The results of the pre-run in (2) have to be stored in a form that is available and easily accessible to further model runs. This can be done using most commercial software products. However, it is a rather expensive way to store a 1D profile of deformations in a 2D or 3D array. More important is that the pre-run has to be designed properly. For a relevant steady state it may not be sufficient to only omit the outer forcing terms. Also conditions at vertical boundaries may have to be changed to roller conditions to allow the entire model to reach a steady state. In order to circumvent these problems, the application of strategy (3) may be an alternative.

All mentioned strategies deliver the deformation for a thickness H that would hold for the layer system if gravity is neglected. The height of a system, which is obtained from measurements, however has developed under the influence of gravity (H_0). In order to obtain the corresponding initial thickness H (formula (7)) that has to be used in modelling, the three strategies can be extended. For the mentioned pre-run methods several trial and error runs can be applied to obtain an H, for which the corresponding H_0 is sufficiently good approximated. Using the derived analytical solution, H is obtained using the combined equations (7) and (10):

$$H = \frac{1}{A} \pm \sqrt{\frac{\left(1 - AH_0\right)^2}{A^2} - H_0^2} \qquad (11)$$

No extensions and additional model runs are necessary. The obtained value H has to be used as layer thickness.

One example is shown to illustrate the methodology. Figure 1 shows simulated deformation profiles, modelled using COMSOL Multiphysics. The poroelastic mode was used to model the transition from an initial unstable state to the steady state, which is given by the analytical solution, equation (5). Gravity was considered here as active.
Vertical displacement is shown on the y-axis against height above base on the x-axis. Negative values indicate deformation in direction of gravity. Starting from zero at the fixed base of the deforming layer the absolute values of deformation increase with arc length. The legend depicts time instants, for which curves are depicted. Dots indicate the analytical solution.

The total depth of the system is 3 m, and the deformations to are very small ($<5\bullet10^{-6}$ m). The legend shows times [s] of the transitional states.

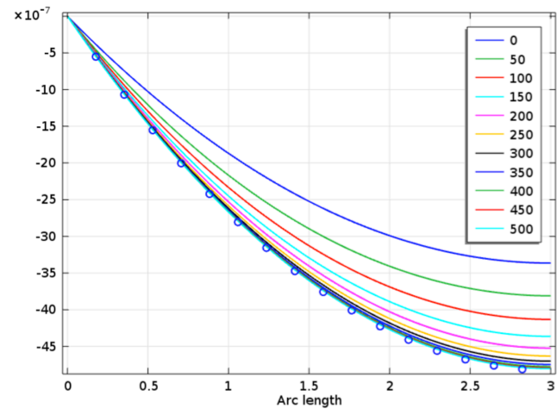

Figure 1. Transient development towards the steady state, given by the derived analytical solution (shown by circles)

Parameter Range Examination

Note that the deformation w in a consolidated system is negative if the vertical space axis is counted positive upwards. Because the last term in equation (5) z-2H is negative, all leading terms in the equation should be positive. This condition gives some limitations concerning the parameter space.

The density term -($\alpha\rho_f$+ρ) is not problematic as the density of the porous system ρ=$\theta\rho_f$+ρ_b is bigger than the fluid density (ρ_b denotes the bulk density). The coefficient α<1 even reduces the smaller term.

The three terms including the Poisson ratio provide a positive value, at least for the most important range of Poisson ratios 0<ν<0.5. Figure 2 depicts the value of the three factors including ν in formula (6) combined, i.e. the function $(1+\nu)(1-2\nu)/(1-\nu)$.

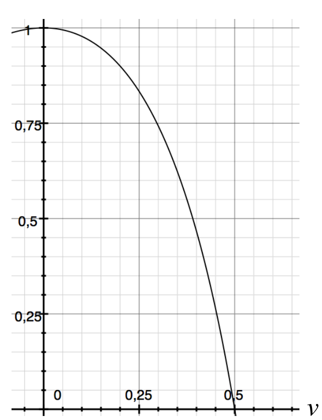

Figure 2. The combined effect of the terms including the Poisson ratio on the maximum deformation

For low values of v the shown function lies near to 1 and decreases rapidly towards 0 for v>0.25.

In order to further study the limits of the given approach concerning the included parameter, a reference parameter set is defined. The values are given in Table 1. These values are used, if not noted otherwise. With these parameters for A the value of $1.43 \cdot 10^{-5}$ is obtained.

Table 1: Reference parameter set

Parameter	Value [unit]	Parameter	Value [unit]
Young modulus E	100 [MPa]	Biot parameter α	1
Poisson ratio ν	0.25	Porosity θ	0.25
Fluid density ρ_f	1000 [kg/m^3]	Gravity accelera-tion g	9.81 [m/s^2]
Bulk density ρ_b	2500 [kg/m^3]	Thickness H	3000 [m]

For the reference parameter set Figure 3 shows the maximum deformation relative to the initial height H_0, according to formula (10). Minimum geostatic thickness was chosen as 1. If the thickness H_0 exceeds the maximum depicted value, 3500 m, the radicants in equations (10) and (11) become negative and the square roots become imaginary and there exist no non-imaginary solutions anymore. The limit is obviously given by the condition

$$2AH_0 = 1 \qquad (12)$$

The maximum deformation w_{max} must be smaller than the height H_0, or $w_{max}/H_0<1$. The maximum deformation cannot exceed the layer thickness.

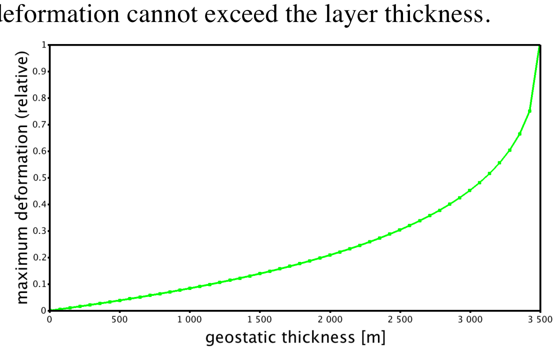

Figure 3. Maximum deformation (normalized) in dependence of thickness H_0 for the reference parameter set

Figure 3 shows, that for higher values of the geostatic thickness the relative maximum deformation increases. In fact it increases both in absolute and relative values. Moreover, the graph shows that the solution delivers reasonable values for layer thicknesses up to 3500 m. Thus the derived solution obviously delivers appropriate values for real geological applications because this value will hardly be exceeded in any application.

Figure 4 depicts the relative deformation w_{max}/H_0 in dependence on initial thickness H for three different values of v. All other parameters are selected according to the reference case parameter list, presented in Table 1. On the abscissa the thickness of the layer is depicted, which ranges up to 12 km. This value is in the range of the thickness of the earth crust, and surely exceeds values of practical interest. Graphs for three different Poisson ratios are presented, which show the relative maximum deformation, as already described in the explanation of Figure 3.

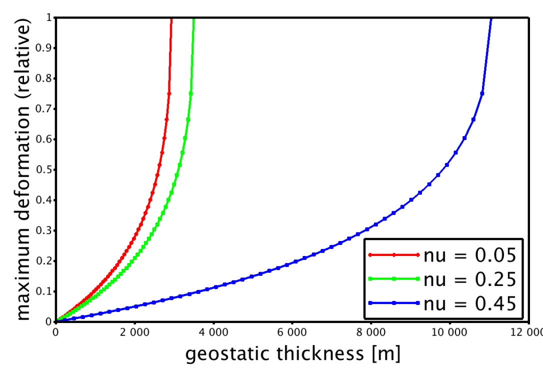

Figure 4. Maximum deformation (normalized) in dependence of thickness H_0 and Poisson ratio v

For increasing values of v the relative maximum deformation decreases. Correspondingly the limit of applicability of the derived approach is reached for higher values of H_0. As could be expected from the findings of Figure 2, the deviances are small for low values of v and higher for values near the upper range of v=0.5.

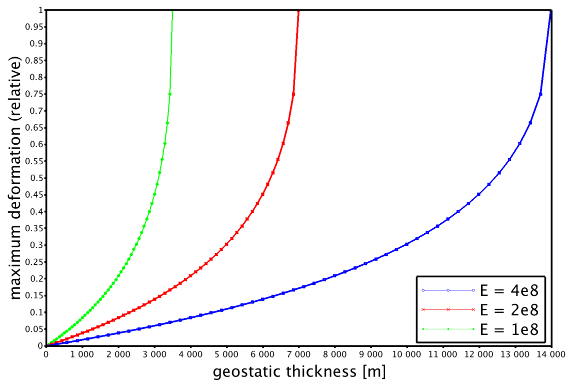

Figure 5. Maximum deformation (normalized) in dependence of thickness H_0 and Young modulus E

In Figure 5 the effect of the Young modulus is visualized similarly to Figure 4. The figure shows that with increasing value for E the relative deformation is decreasing. Correspondingly the limit of applicability, where w_{max}/H_0 becomes unity, is reached for higher geostatic thicknesses. For the highest Young modulus (E=4•10^8 MPa) that limit is reached for a layer thickness of 14 km; for the medium Young modulus the corresponding value is 7 km, and for the lowest (E=10^8 MPa) it is 3.5 km.

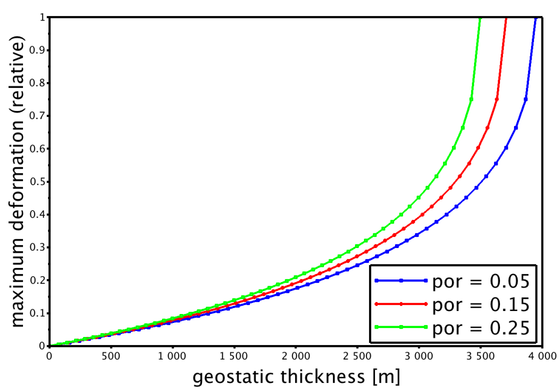

Figure 6. Maximum deformation (normalized) in dependence of thickness H_0 and porosity θ

In Figure 6 the effect of porosity is visualized similarly to Figures 4 and 5. The figure shows that with increasing value for θ the relative deformation is increasing. Correspondingly the limit of applicability, where w_{max}/H_0 becomes unity, is reached for lower geostatic thicknesses.

Conclusions

An analytical solution was derived for the geostatic steady state under influence of gravity. The formula delivers a deformation profile that correspondents with the hydrostatic profile of pore pressure. We extending the study to deliver expressions, connecting the height of an unconsolidated layer without action of gravity, maximum deformation and the thickness of a layer, consolidated under the influence of gravity.

The presented formula can be utilized for various purposes, such as

- analytical solution for steady states
- initial state in unsteady simulations
- boundary condition at vertical edges
- comparison of steady states
- comparison of steady and unsteady states
- benchmark for code developers

It was shown that the proposed approach using the derived analytical formula is valid for the usual parameter range of real geological applications.

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