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Implementation of the Isotropic Linear Cosserat Models based on the Weak Form

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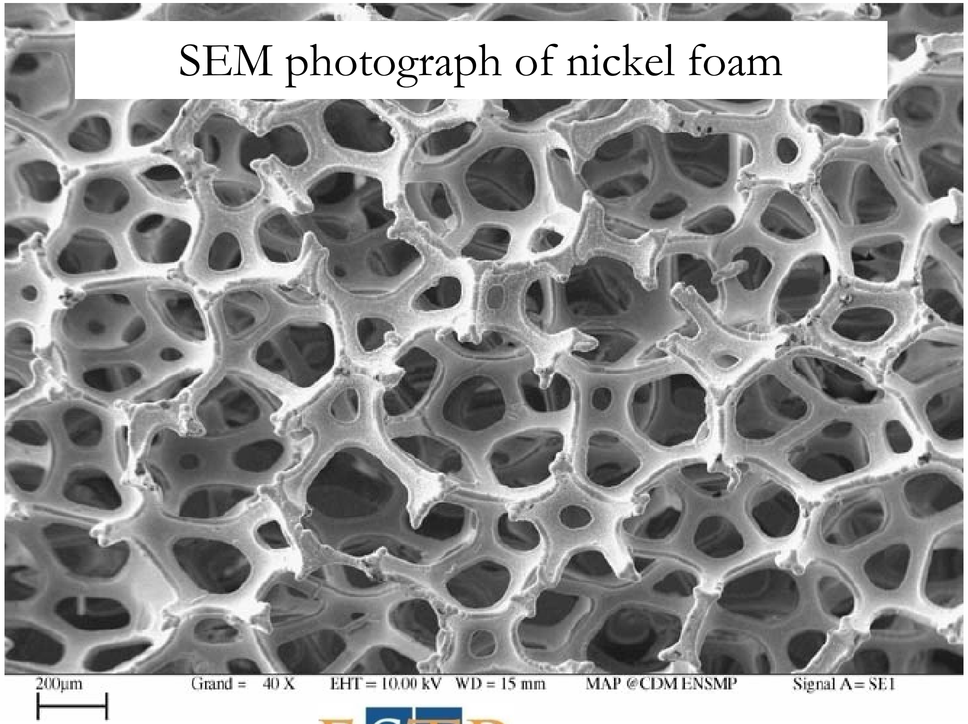
Theory of the extended continuum media

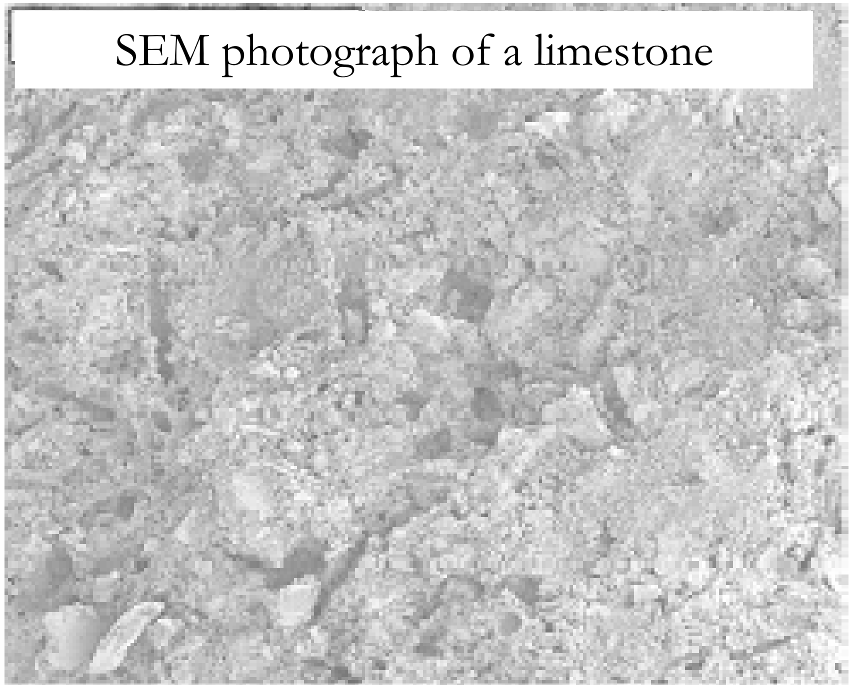
- General comparison among the extended continuum theories

Name	No. of DOFs	References
Cauchy	3	Cauchy (1823)
Micro dilatation	4	Cowin (1971)
Cosserat	6	Kafadar-Eringen (1976)
Microstretch	7	Forest-Sievert (2006)
Microstrain incompressible	8	-
Microstrain	9	-
Incompressible Micromorphic	11	-
Micromorphic	12	Eringen, Midlin (1964)

Cosserat medium as an explicit size effect approach

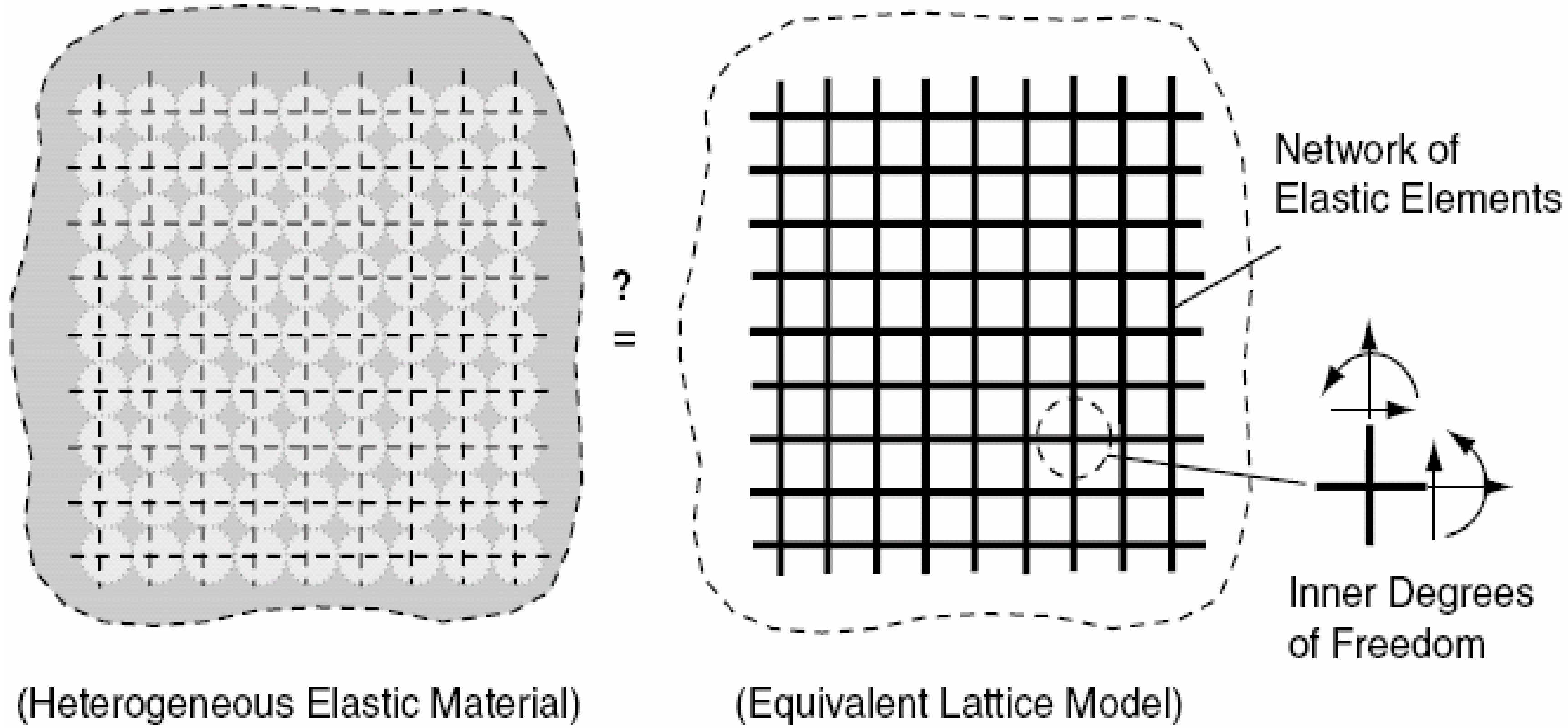
- What are the heterogeneous materials?
  - **Nickel foams** (energy saving), **Natural limestone** (stone weathering prediction), **Ceramic materials** (metallurgical applications)





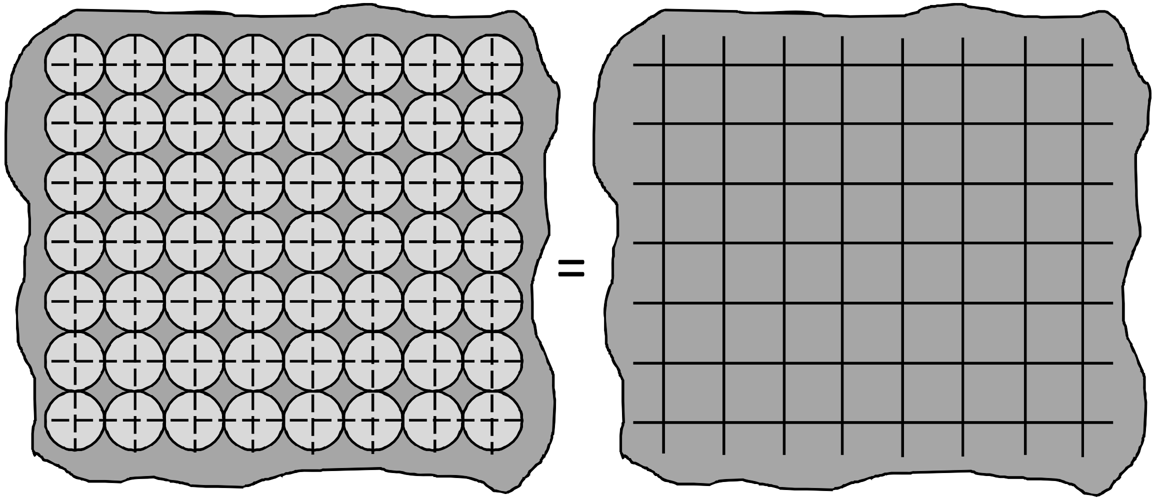
Cosserat medium as an explicit size effect approach

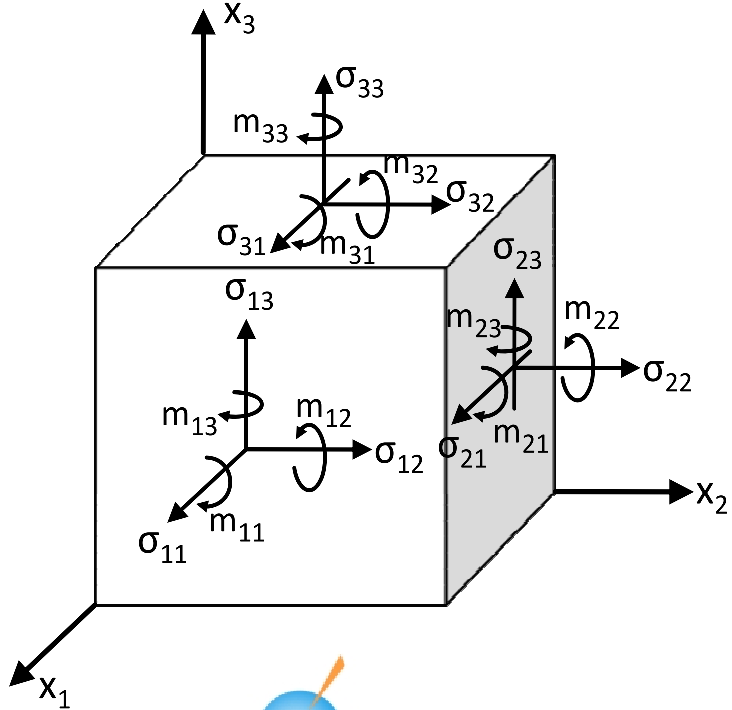
- How to deal with the heterogeneous materials?



Cosserat medium as an explicit size effect approach

- The Cosserat ideas is to introduce a phenomenological micro-rotation parameter.
- The micro-rotations can easily present the rotations in the equivalent lattice model.





Why Cosserat medium?

- Both **linear** and **non-linear Cosserat** methods are capable of treating the size effects in an explicit manner via the **characteristic lengths** assumptions,
- The characteristic lengths have been assumed to be defined by the **materials microstructure**,
- The size effects can be readily obtained via the Cosserat assumptions (**smaller specimens are stiffer than larger ones**).
- The **micro-rotations** for the material microstructure have been computed **directly** from the numerical simulations.

Linear Cosserat elasticity formulation

- The energy minimization for the static case with respect to the displacement and micro-rotation vector:

$$I(\bar{u},\bar{\bar{A}}) = \overbrace{\int_{\Omega}^{} \bar{\bar{\sigma}}:\bar{\bar{\varepsilon}}\,dV}^{Strain\ energy} + \overbrace{\int_{\Omega}^{} \bar{\bar{m}}:\bar{\bar{k}}\,dV}^{Curvature\ energy} - \overbrace{\int_{\partial\Omega}^{} \overline{t^{(n)}}\cdot\bar{u}\,dS}^{Stress\ vector\ work} - \overbrace{\int_{\partial\Omega}^{} \overline{Q^{(n)}}\cdot\bar{u}\,dS}^{Couple\ stress\ vector\ work}$$

$$- \overbrace{\int_{\Omega}^{} \rho\bar{b}\cdot\bar{u}\,dV}^{body\ force\ work} - \overbrace{\int_{\Omega}^{} \rho\bar{c}\cdot\bar{u}\,dV}^{body\ moment\ work} \mapsto w.r.t.\ \bar{u}\ and\ \bar{\bar{A}}$$

$$\mathrm{Div}\,\bar{\bar{\sigma}}^T + \rho\bar{b} = 0 \iff \sigma_{ji,j} + \rho b_i = 0$$

$$\mathrm{Div}\,\bar{\bar{m}}^T + \rho\bar{c} = -4\mu_c\,axl(skew\,\bar{\bar{\epsilon}}) \iff \mathrm{Div}\,\bar{\bar{m}}^T + \rho\bar{c} = \bar{\bar{\bar{e}}}:\bar{\bar{\sigma}} \iff m_{ji,j} + \rho c_i = e_{ijk}\sigma_{jk}$$

Linear Cosserat elasticity formulation-basic equations

- Kinematics:

$$\bar{\bar{\epsilon}} = \nabla \otimes \bar{u} - \bar{\bar{A}} \iff \bar{\bar{\epsilon}} = \nabla \otimes \bar{u} - \text{anti}\,(\bar{\varphi}) \iff \epsilon_{ij} = u_{j,i} - e_{ijk}\varphi_k$$

- The constitutive laws:

$$\bar{\bar{\sigma}} = \lambda\, tr(\bar{\bar{\epsilon}})\bar{\bar{\mathbb{I}}} + 2\mu \text{ sym } \bar{\bar{\epsilon}} + 2\mu_c \text{ skew } \bar{\bar{\epsilon}} \iff \sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + \mu\left(\epsilon_{ij} + \epsilon_{ji}\right) + \mu_c\left(\epsilon_{ij} - \epsilon_{ji}\right)$$

$$\bar{\bar{m}} = \alpha \text{ tr}(\nabla \otimes \bar{\varphi})\bar{\bar{\mathbb{I}}} + \beta\,(\nabla \otimes \bar{\varphi})^{\mathrm{T}} + \gamma\, \nabla \otimes \bar{\varphi} \iff \bar{\bar{m}} = \alpha \text{ tr}\left(\bar{\bar{k}}\right)\bar{\bar{\mathbb{I}}} + \beta\, \bar{\bar{k}}^T + \gamma\, \bar{\bar{k}} \quad where \quad \bar{\bar{k}} = \nabla \otimes \bar{\varphi}$$

$$\iff m_{ij} = \alpha\, \varphi_{k,k}\delta_{ij} + \beta\, \varphi_{i,j} + \gamma\, \varphi_{j,i} \iff m_{ij} = \alpha k_{kk}\delta_{ij} + \beta\, k_{ji} + \gamma\, k_{ij}$$

- Equilibrium equations:

$$\text{Div } \bar{\bar{\sigma}}^T + \rho \bar{b} = 0 \iff \sigma_{ji,j} + \rho b_i = 0$$

$$\text{Div } \bar{\bar{m}}^T + \rho \bar{c} = -4\mu_c\, axl(skew\, \bar{\bar{\epsilon}}) \iff \text{Div } \bar{\bar{m}}^T + \rho \bar{c} = \bar{\bar{\bar{e}}} : \bar{\bar{\sigma}} \iff m_{ji,j} + \rho c_i = e_{ijk}\sigma_{jk}$$

Non-linear Cosserat elasticity formulation-basic equations

- **Kinematics:**

$$\bar{\bar{\epsilon}} = \nabla \otimes \bar{u} - \frac{\sin(\|\bar{\varphi}\|)}{\|\bar{\varphi}\|}\bar{\bar{A}} + \frac{1-\cos(\|\bar{\varphi}\|)}{\|\bar{\varphi}\|^2}\bar{\bar{A}}\bar{\bar{A}} - \frac{\sin(\|\bar{\varphi}\|)}{\|\bar{\varphi}\|}\bar{\bar{A}}\,(\nabla \otimes \bar{u}) + \frac{1-\cos(\|\bar{\varphi}\|)}{\|\bar{\varphi}\|^2}\bar{\bar{A}}\bar{\bar{A}}(\nabla \otimes \bar{u})$$

- The constitutive laws:

$$\bar{\bar{\sigma}} = \lambda\, tr(\bar{\bar{\epsilon}})\bar{\bar{\mathbb{I}}} + 2\mu \,\mathrm{sym}\, \bar{\bar{\epsilon}} + 2\mu_c \,\mathrm{skew}\, \bar{\bar{\epsilon}} \Leftrightarrow \sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + \mu\left(\epsilon_{ij} + \epsilon_{ji}\right) + \mu_c\left(\epsilon_{ij} - \epsilon_{ji}\right)$$

$$m_{ij} = \alpha\, \varphi_{k,k}\delta_{ij} + \beta\, \varphi_{i,j} + \gamma\, \varphi_{j,i} \Leftrightarrow m_{ij} = \alpha k_{kk}\delta_{ij} + \beta\, k_{ji} + \gamma\, k_{ij}$$

- Equilibrium equations:

$$\mathrm{Div}\, \bar{\bar{\sigma}}^T + \rho \bar{b} = 0 \Leftrightarrow \sigma_{ji,j} + \rho b_i = 0$$

$$\mathrm{Div}\, \bar{\bar{m}}^T + \rho \bar{c} = -4\mu_c\, axl(skew\, \bar{\bar{\epsilon}}) \Leftrightarrow \mathrm{Div}\, \bar{\bar{m}}^T + \rho \bar{c} = \bar{\bar{\bar{e}}} : \bar{\bar{\sigma}} \Leftrightarrow m_{ji,j} + \rho c_i = e_{ijk}\sigma_{jk}$$

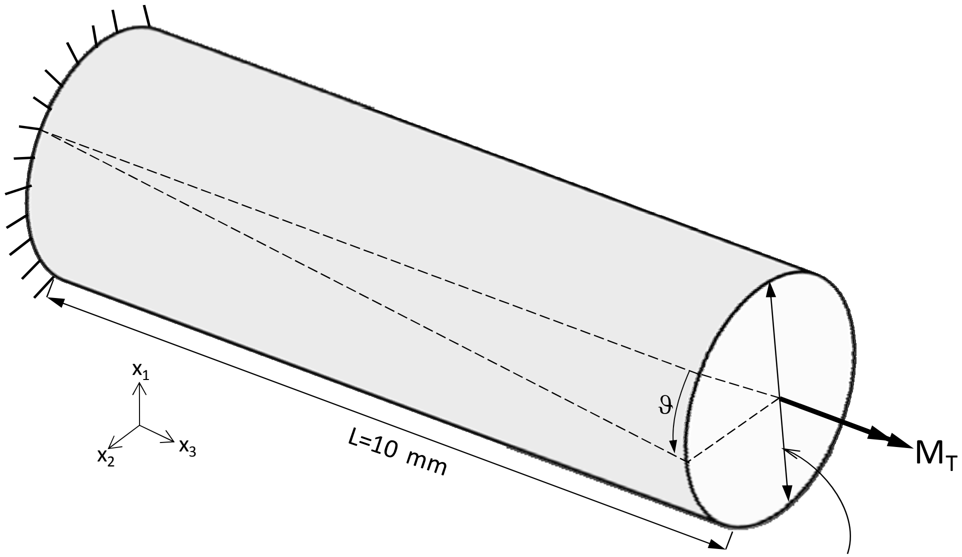
FE implementation of the Cosserat media

- Why Comsol Multiphysics?

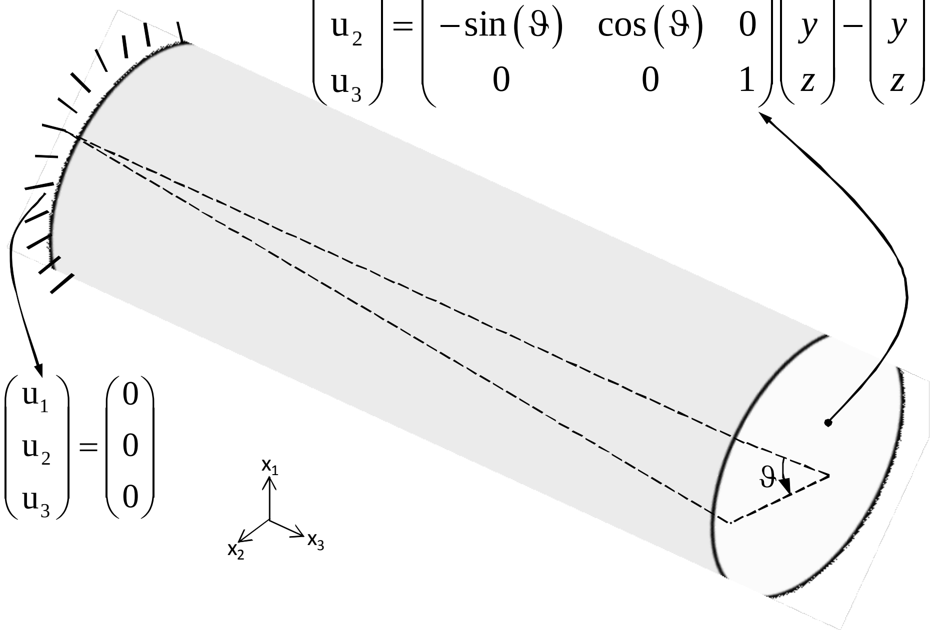
FEM Codes	Convergences Problem		Descriptions
	Linear Cosserat	Non-linear Cosserat	
FlexPDE	Yes	Not-tested	The available solver doesn't support this type of equations…
FEAP	No	No	No. of DOFs is limited and there is no parallel solver….
Abaqus	No	Not-tested	Compilation problem for non-linear Cosserat under C++, Intel Fortran
Comsol MP 3.4	**No**	**No**	The Pardiso solver (parallel solver) enables us to compute more than 704616 DOFs (25056 elements for 3D case)

FE implementation of the Cosserat media

- Geometry of the model:
- Applied BCs



Young's modulus (MPa)	Poisson's ratio (-)
1E6	0.30

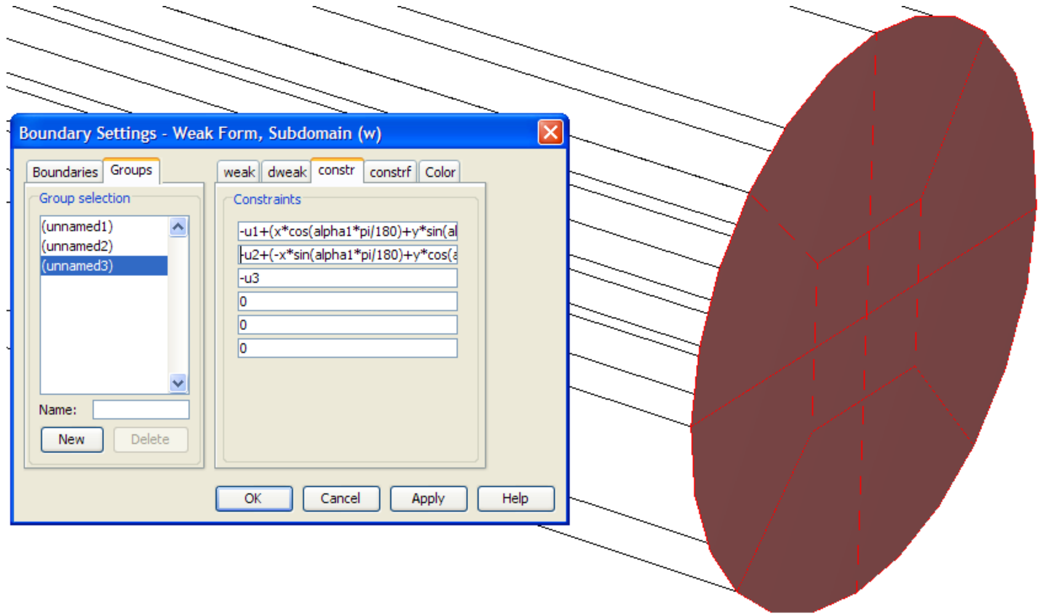


$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) & 0 \\ -\sin(\vartheta) & \cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

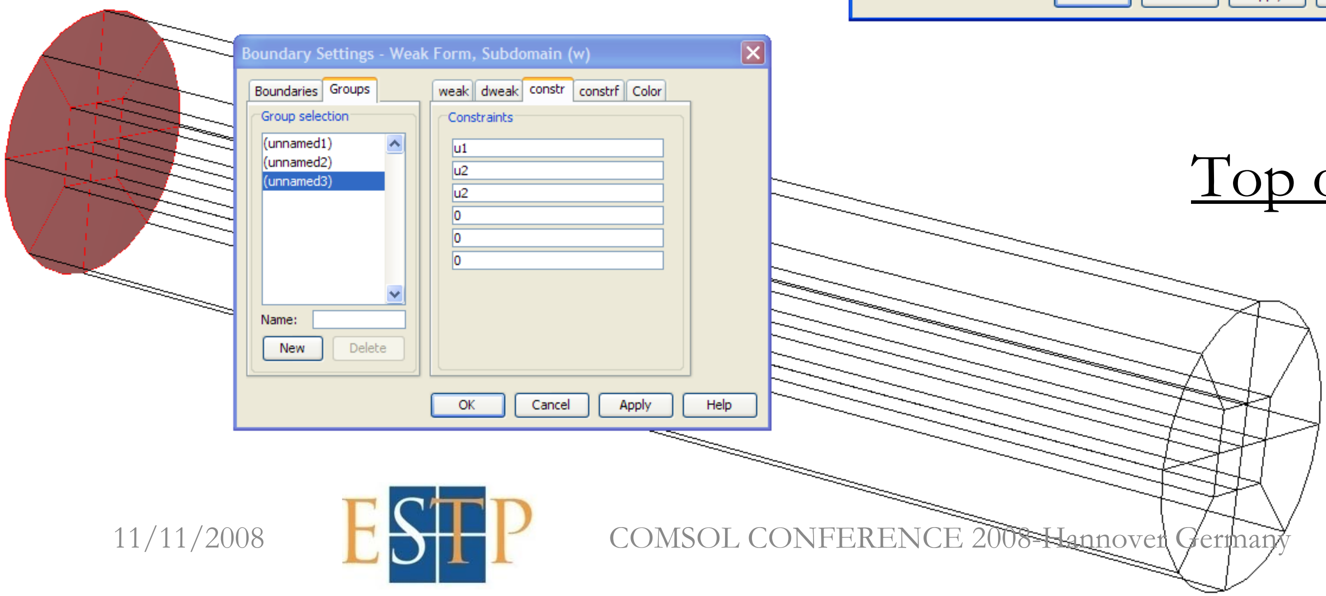
$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

FE implementation of the Cosserat media

- Applied BCs



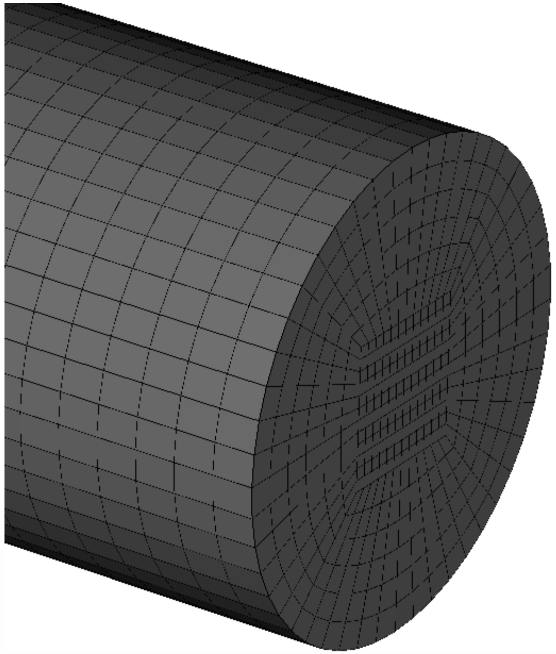
Bottom of cylindrical bar



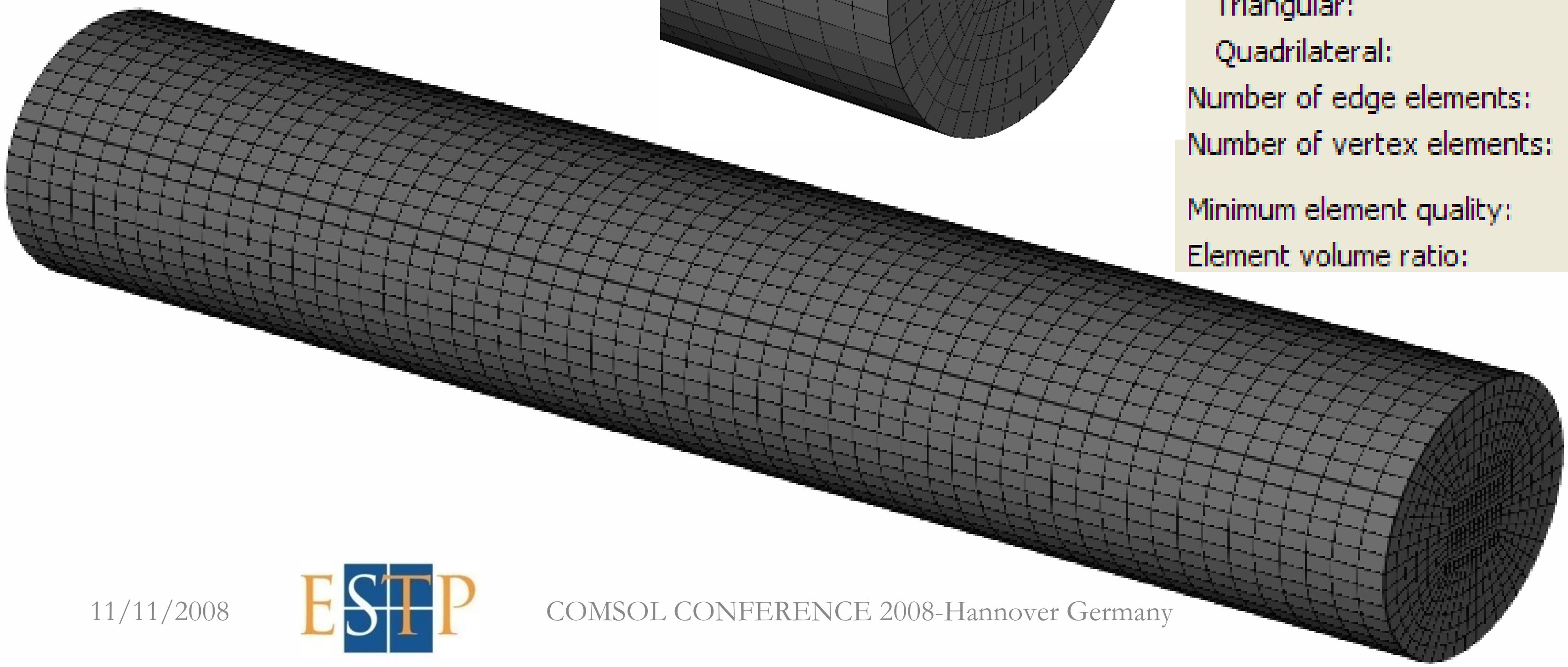
Top of cylindrical bar

FE implementation-Mesh density

- Mesh statistics:



Extended mesh:	
Number of degrees of freedom:	704616
Base mesh:	
Number of mesh points:	26963
Number of elements:	25056
Tetrahedral:	0
Prism:	0
Hexahedral:	25056
Number of boundary elements:	10608
Triangular:	0
Quadrilateral:	10608
Number of edge elements:	1322
Number of vertex elements:	34
Minimum element quality:	0.3461
Element volume ratio:	0.1578

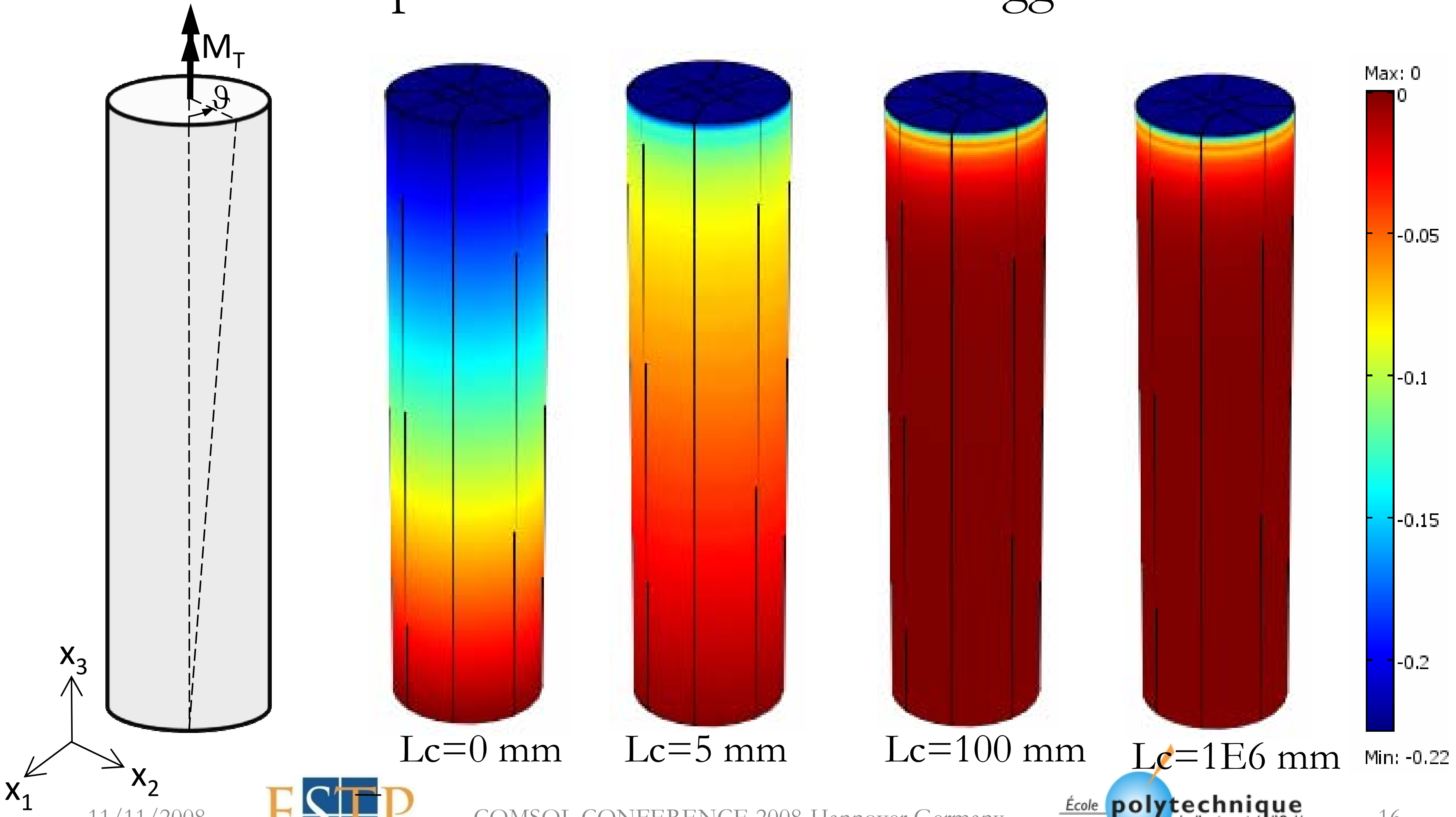


Difficulties and remedies!!

- PDE General form/PDE Coefficient form or **PDE Weak form**:
  - The PDE general form et/or PDE Coefficient are not suitable tools for the coupled numerical models (**we lost 3 months!!!!**).
  - PDE Weak form is the **best choice** and it is tested in this study.
  - Ansatz **shape functions** have been applied into the FE analyses (**Quadratic** and **linear** shape functions for **displacement** and **micro-rotation vectors**)

Numerical results-Macro-rotation in x_3 direction

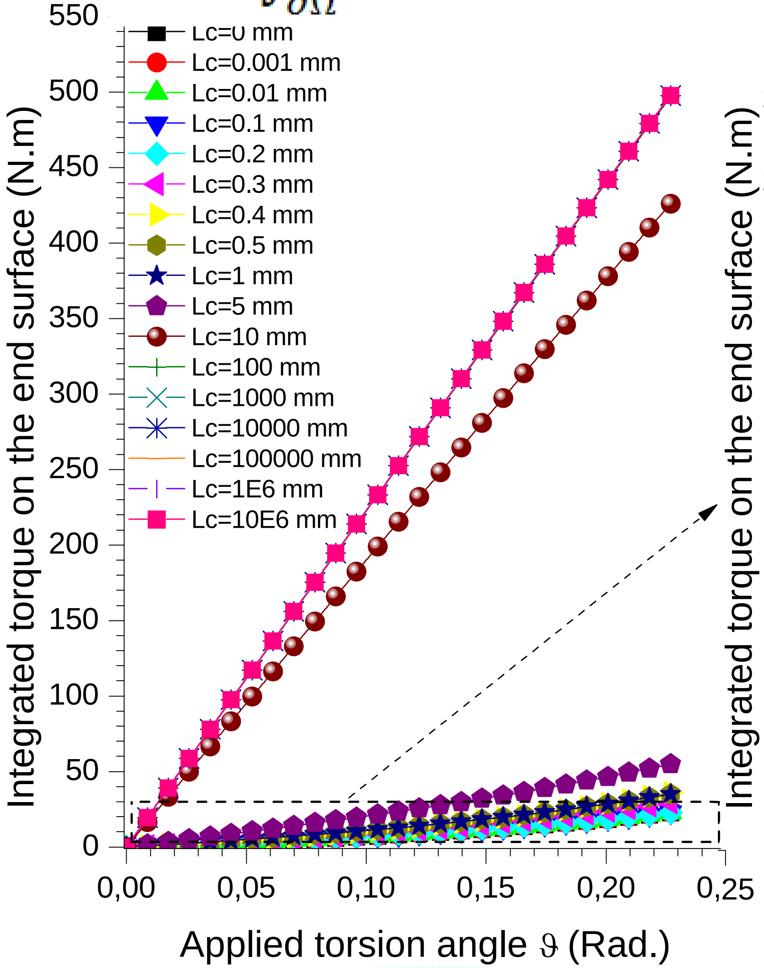
- The smaller specimens are stiffer than bigger ones.

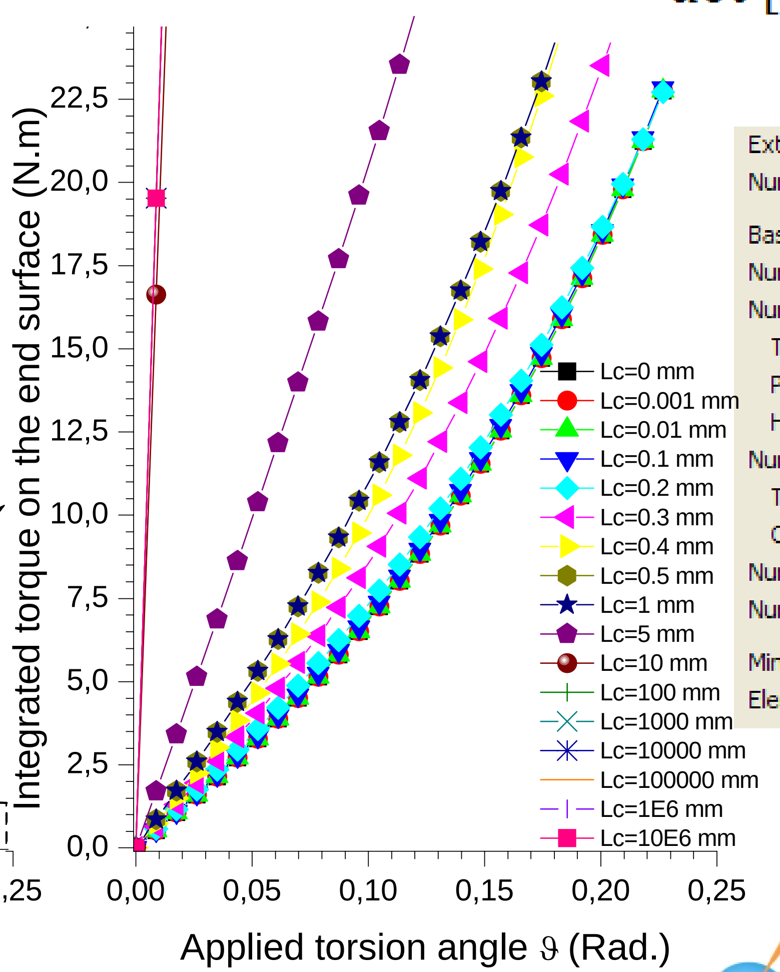


Lc=0 mm Lc=5 mm Lc=100 mm Lc=1E6 mm

Numerical results and verifications (Geometrically exact case)

$$M_T = \int_{\partial\Omega} (x_1 \sigma_{32}^{exact} - x_2 \sigma_{31}^{exact}) dx_1 dx_2 \qquad \bar{\bar{\sigma}}^{exact} = \frac{1}{\det[\bar{\bar{F}}]} \bar{\bar{\sigma}}^{Cauchy} \bar{\bar{F}}^T$$

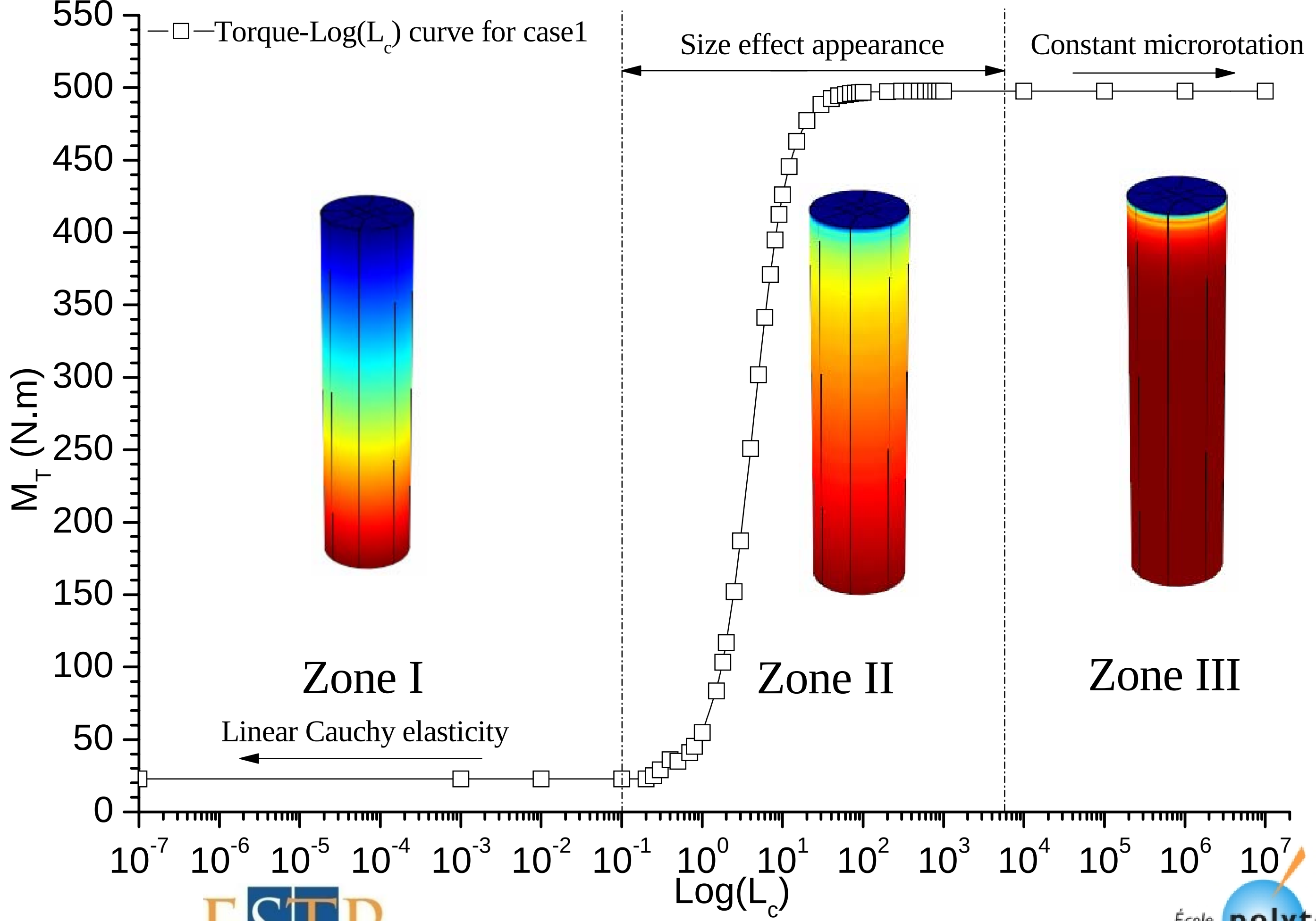


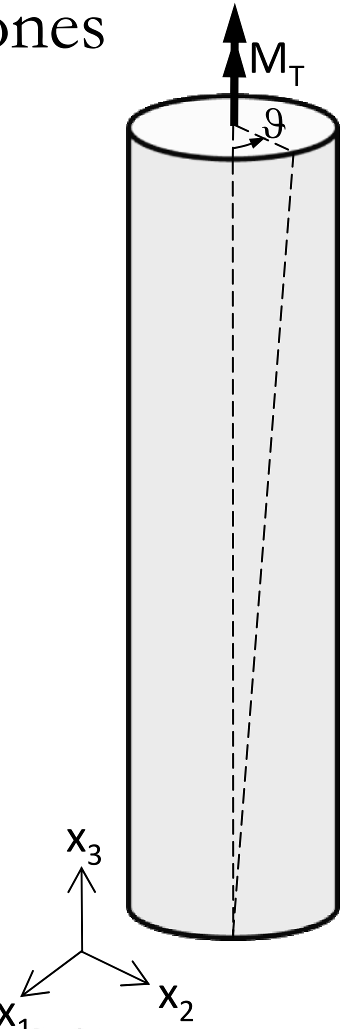


Extended mesh:	
Number of degrees of freedom:	298878
Base mesh:	
Number of mesh points:	6655
Number of elements:	5832
Tetrahedral:	0
Prism:	0
Hexahedral:	5832
Number of boundary elements:	4752
Triangular:	0
Quadrilateral:	4752
Number of edge elements:	1086
Number of vertex elements:	34
Minimum element quality:	0.7516
Element volume ratio:	0.1706

Numerical results-Semi-logarithmic diagram

- The smaller specimens are stiffer than bigger ones



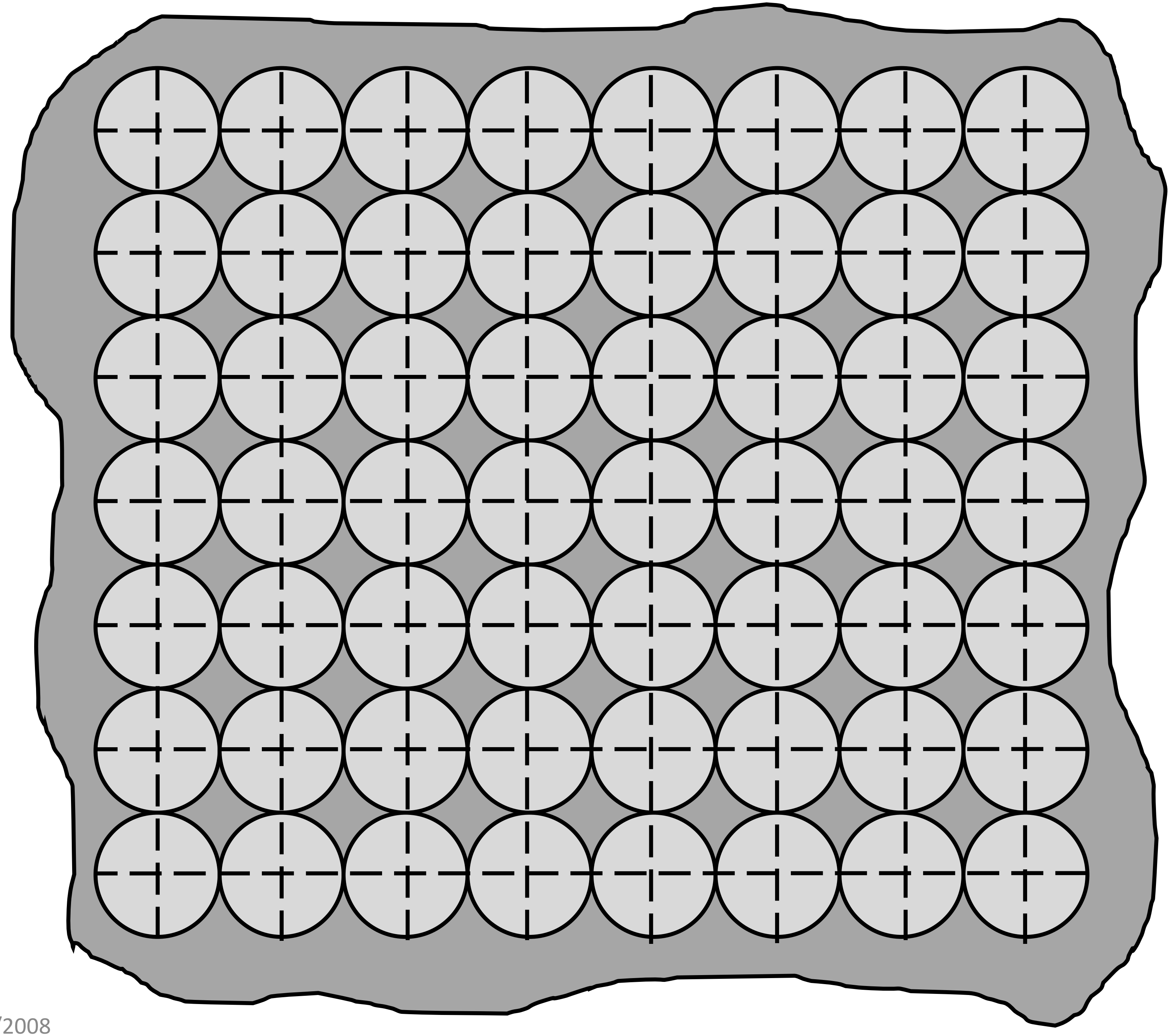


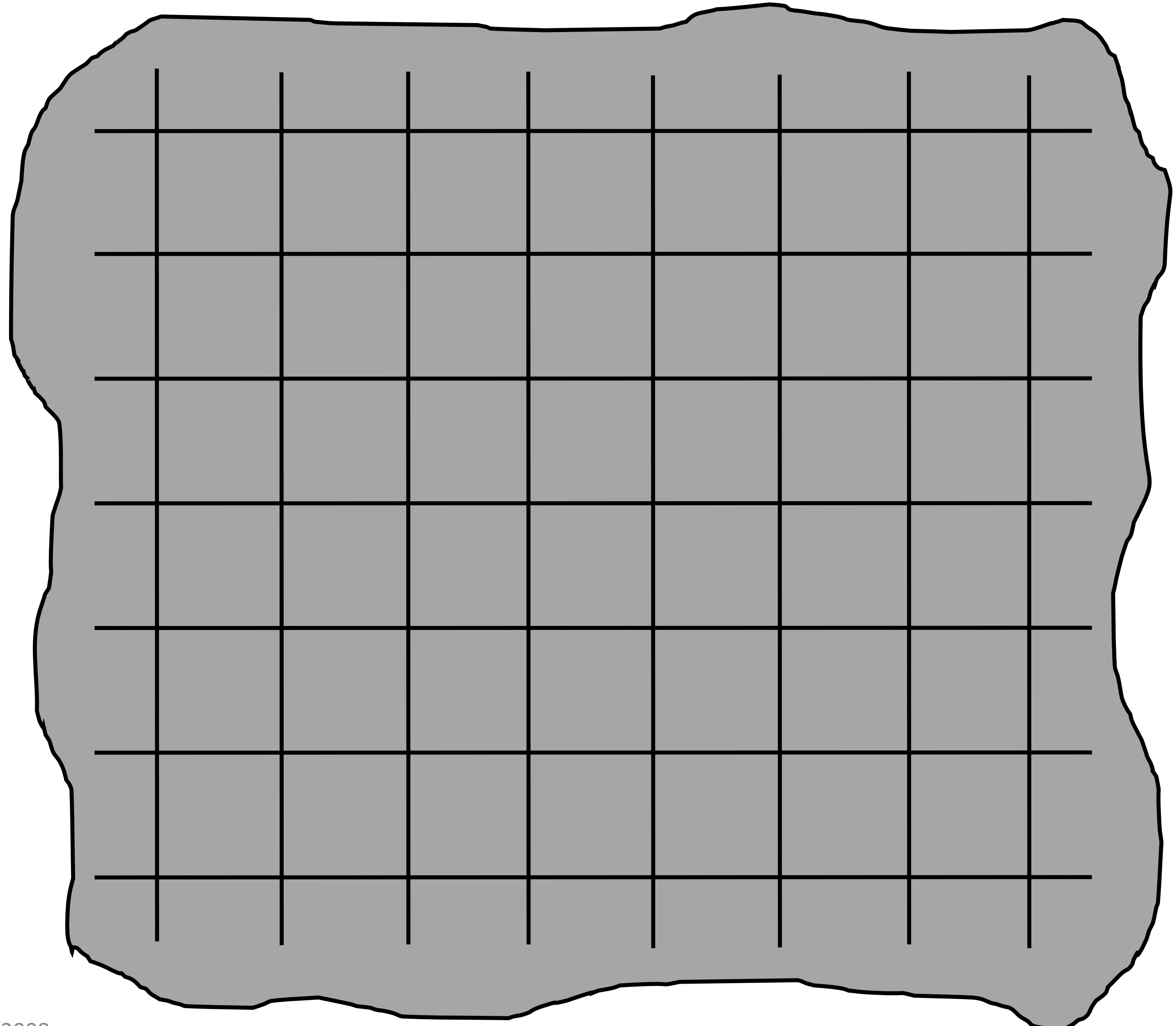
Syntheses and conclusions

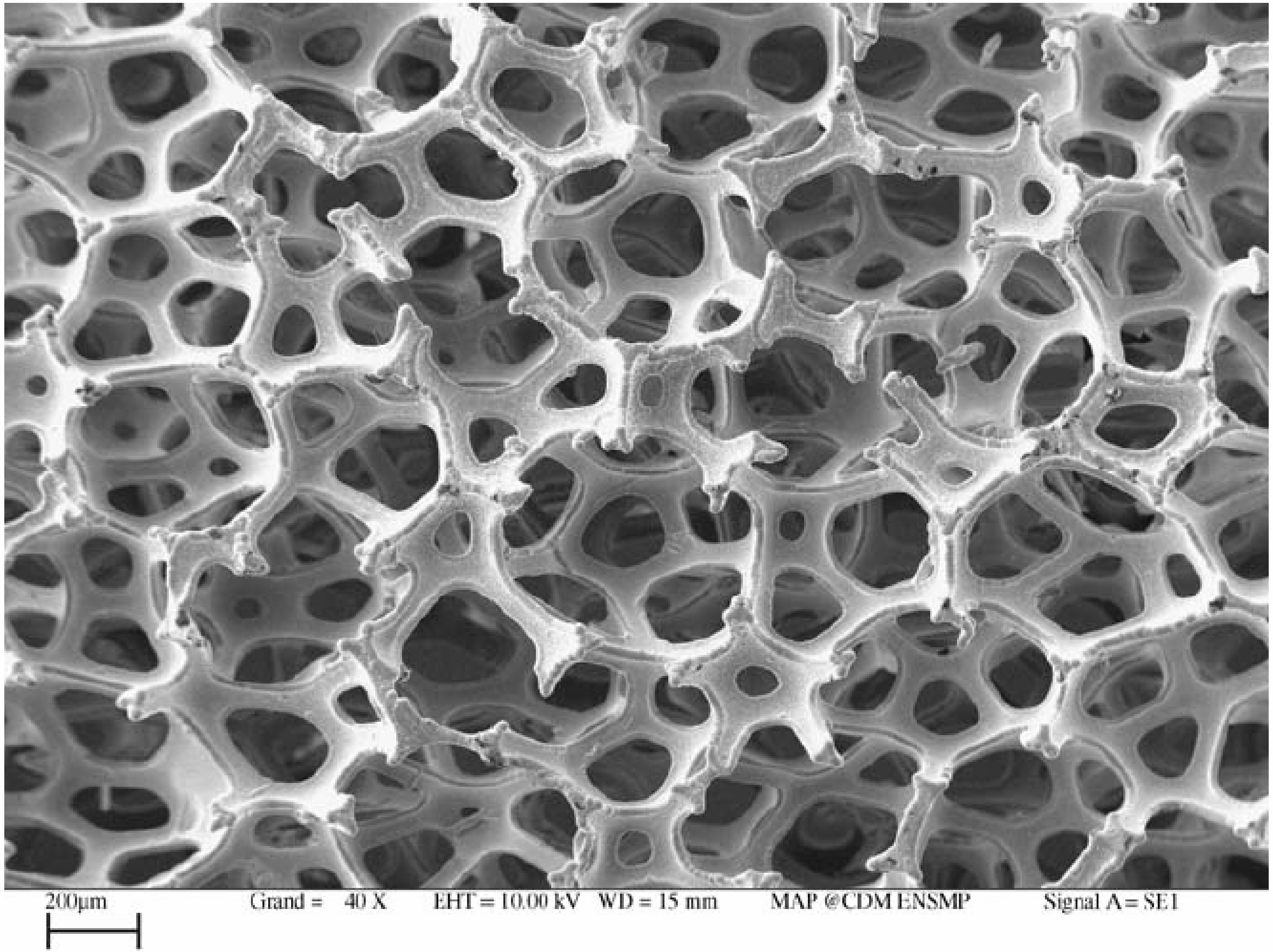
- The linear Cosserat elasticity has been successfully implemented into COMSOL MP 3.4,
- The non-linear Cosserat elasticity has been equally implemented into COMSOL MP 3.4 (under verification…),
- The size effect has been found out for different L_c values,
- No artificial BCs has been applied in the calculations and all micro-rotations have been calculated by FEM during computation stage,
- The mesh dependency problem was removed using Ansatz shape functions (quadratic and linear for the displacement and micro-rotation vectors, respectively),
- The outcomes for the cylindrical bar can be illustrated in a semi-logarithmic diagram and three distinct zones can be easily distinguished,
- Zone I presents no size effects (large specimens),
- Zone II presents the size effects (smaller specimens are stiffer than larger ones),
- Zone III deals with the constant micro-rotation case in which the curvature energy vanishes.

Outlooks and future plans

- The non-linear Cosserat numerical models verification by means of the analytical solutions (Boundary value problem and conformal solution),
- The implementation of the elastic-plastic linear/non-linear Cosserat models under COMSOL MP environment,
- The micro-morphic models implementation into COMSOL MP (there are 12 unknown state variables instead of 6 variables for the linear and non-linear Cosserat models),
- The results verification by means of the analytical solutions of boundary value problem,
- The implementation of the elastic-plastic linear/non-linear micro-morphic models under COMSOL MP environment,
- The utilization of new COMSOL MP (Comsol 3.5) and enjoying new enhancements!!






200µm
Grand = 40 X
EHT = 10.00 kV
WD = 15 mm
MAP @CDM ENSMP
Signal A = SE1