COMSOL CONFERENCE
2016 BOSTON

Equation-Based Modeling: Simulation of a Flow With Concentrated Vorticity in an Unbounded Domain

John M. Russell, Professor Emeritus
Florida Inst. of Technology, Melbourne, Florida

COMSOL Conference, Boston, October 6, 2016

Objectives

Develop a COMSOL simulation such that:

1. The physical domain is three-dimensional and has no exterior boundary;
2. The spin tensor, $\nabla\mathbf{u} - (\nabla\mathbf{u})^T$, (in which $\mathbf{u}$ is the velocity field) differs from the null tensor only in a bounded subdomain;
3. An analytical solution is available for comparison.

Physical Domains; Features of Hill's Spherical Vortex

Let $\mathcal{R}^i$ and $\mathcal{R}^e$ denote the three-dimensional regions interior and exterior to a sphere of radius a, respectively. Let the motion be solenoidal everywhere, so $\operatorname{div}\mathbf{u} = 0$ identically. Let the motion in $\mathcal{R}^e$ be uniformly irrotational. Let the motion in $\mathcal{R}^i$ be rotational with

$$\nabla\mathbf{u} - (\nabla\mathbf{u})^T = A(\mathbf{r}\otimes\hat{\mathbf{i}}_3 - \hat{\mathbf{i}}_3\otimes\mathbf{r}) \ ,$$

in which $\hat{\mathbf{i}}_3$ is a constant vector, A is a constant scalar, and $\mathbf{r}$ denotes position relative to the center of the sphere. Notation: $(\mathbf{a}\otimes\mathbf{b})\mathbf{v} := \mathbf{a}(\mathbf{b}\bullet\mathbf{v})$.

Variational principles in $\mathcal{R}^i$ and $\mathcal{R}^e$

Consider the problem of minimizing F, in which

$$F := \iiint\limits_{\mathcal{R}\in\{\mathcal{R}^i,\mathcal{R}^e\}} \{(\operatorname{div}\mathbf{u})^2 + \tfrac{1}{4}\|\nabla\mathbf{u} - (\nabla\mathbf{u})^T - W]\|^2\}\, dV \ .$$

Here

$$W := \begin{cases} A(\mathbf{r}\otimes\hat{\mathbf{i}}_3 - \hat{\mathbf{i}}_3\otimes\mathbf{r}), & \text{for all } \mathbf{r}\in\mathcal{R}^i \ , \\ O, & \text{for all } \mathbf{r}\in\mathcal{R}^e \ , \end{cases}$$

in which O in the last line denotes the null tensor.

NON UNIQUENESS OF THE VELOCITY FIELD, u, THAT MINIMIZES F

Note that F attains the minimium value 0 for any function $\mathbf{u}$ that satisfies the system of equations

$$\operatorname{div}\mathbf{u} = 0 \quad , \quad \nabla\mathbf{u} - (\nabla\mathbf{u})^T = W \; .$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are any two solutions of this system then

$$\operatorname{div}(\mathbf{u}_2 - \mathbf{u}_1) = 0 \; , \; \nabla(\mathbf{u}_2 - \mathbf{u}_1) - [\nabla(\mathbf{u}_2 - \mathbf{u}_1)]^T = O.$$

Any function of the form $\mathbf{u}_2 - \mathbf{u}_1 = \nabla\phi$, satisfies this system provided $\nabla^2\phi = 0$.

A BOUNDARY CONDITION THAT REMOVES THE AMBIGUITY IN $\mathbf{u}$

In potential theory if a solution of $\nabla^2\phi = 0$ (in a simply connected domain) satisfies the NEUMANN boundary condition

$$\nabla\phi \bullet \hat{\mathbf{n}} = 0$$

then $\nabla\phi = \mathbf{0}$ throughout that domain. Thus, in the present problem $\mathbf{u}_2 - \mathbf{u}_1 = \nabla\phi$ will be vanish if $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{n}} = 0$, which will be the case if $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy a common *normal velocity* condition $\mathbf{u}_i \bullet \hat{\mathbf{n}} = m$, $i \in \{1, 2\}$, in which m is a given function of position on the boundary.

Variational formalism in $\mathcal{R}^i$, I

If one starts with the definition of F then, in the terminology of the calculus of variations, one may derive an identity satisfied by *the first variation*, δF of F, namely

$$\delta F + \iint_{\partial\mathcal{R}^i} \delta\mathbf{u} \bullet [-\Gamma(\hat{\mathbf{n}})] dA + \iiint_{\mathcal{R}^i} \delta\mathbf{u} \bullet \operatorname{div}\Gamma \, dV = 0 \,,$$

in which

$$\Gamma := 2(\operatorname{div}\mathbf{u})I + \nabla\mathbf{u} - (\nabla\mathbf{u})^T - W \,.$$

Variational formalism in $\mathcal{R}^i$, II

If δF vanishes (*i.e.* F is stationary) for arbitrary variations, $\delta \mathbf{u}$ in $\mathcal{R}^i$ and arbitrary tangential components of $\delta \mathbf{u}$ on $\partial \mathcal{R}^i$ one deduces the *natural boundary condition* and Euler-Lagrange equation, respectively, namely

$$-\Gamma(\hat{\mathbf{n}}) = \mathbf{0} \quad , \quad \operatorname{div} \Gamma = \mathbf{0} \, .$$

Modified Kelvin inversion, I

Let a be a given length scale. Let $\mathbf{r}$ denote position relative to an origin, and let $r = |\mathbf{r}|$. Consider the change of position variable $\mathbf{r} \to \mathbf{q}$ defined by the rule $\mathbf{r}/r = -\mathbf{q}/q$ with $rq = a^2$ and $q = |\mathbf{q}|$. Then a is the geometric mean of r and q; so if $q < r$ then $q < a < r$.

If, in particular, $\mathbf{r}$ is a point exterior to the sphere $r = a$ then $\mathbf{q}$ is a point interior to that sphere. I will call the sphere $r = a = q$ the *Bounding Sphere.* (Remark: True Kelvin Inversion has $\mathbf{r}/r = \mathbf{q}/q$.)

Modified Kelvin inversion, II

I have already introduced the constant unit vector $\hat{\mathbf{i}}_3$. Now let $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ be a right-handed orthogonal system of unit vectors. One may resolve any vector or tensor into components relative to the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$, *e.g.* $\mathbf{r} = \sum_{i=1}^{3} r_i \hat{\mathbf{i}}_i$ and $\mathbf{q} = \sum_{i=1}^{3} q_i \hat{\mathbf{i}}_i$. Similarly, one my define operators $\nabla_q\,(\,.\,)$ and $\mathrm{div}_q\,(\,.\,)$ whose expansions relative to the coordinates (q_1, q_2, q_3) are analogous to the expansions of $\nabla(\,.\,)$ and $\mathrm{div}\,(\,.\,)$ relative to the coordinates (r_1, r_2, r_3) (all expansions being relative to $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$).

Modified Kelvin inversion, III

The equations introduced thus far define a function $(q_1, q_2, q_3) \mapsto \mathbf{r}$. In this way the list (q_1, q_2, q_3) constitute a system of *curvilinear co-ordinates*. It so happens that this system is *orthogonal* in the sense that

$$(\partial\mathbf{r}/\partial q_i)\bullet(\partial\mathbf{r}/\partial q_j) = 0 \quad \text{for } i \neq j \ .$$

Furthermore the corresponding *scale factors* (h_1, h_2, h_3), whose generic definition is $h_i := \|\partial\mathbf{r}/\partial q_i\|$, $i \in \{1, 2, 3\}$ have the the common value $h_1 = h_2 = h_3 := h = (a/q)^2$.

Modified Kelvin inversion, IV

From the equations given thus far one may show that the differentials $d\mathbf{r}$ and $d\mathbf{q}$ are related by the identity $d\mathbf{r} = h\,Q(d\mathbf{q})$, in which

$$Q := 2(\mathbf{q}/q) \otimes (\mathbf{q}/q) - I \ ,$$

and in which I is the identity tensor. Here Q is an *orthogonal tensor* (*i.e.* $\|Q(\mathbf{a})\| = \|\mathbf{a}\|$ for all $\mathbf{a}$), so $Q^T = Q^{-1}$. Furthermore, the system of unit vectors $\{Q(\hat{\mathbf{i}}_1), Q(\hat{\mathbf{i}}_2), Q(\hat{\mathbf{i}}_3)\}$ is right handed (which would not be the case for true Kelvin Inversion).

Modified Kelvin inversion, V

From the equations given thus far one may derive the following *transformation rules*

$$\operatorname{div}\mathbf{u} = h^{-3}\operatorname{div}_q\left[Q^T(h^2\mathbf{u})\right] ,$$

$$\nabla\mathbf{u} - (\nabla\mathbf{u})^T$$
$$= h^{-2}Q\{\nabla_q\left[Q^T(h\mathbf{u})\right] - \nabla_q\left[Q^T(h\mathbf{u})\right]^T\}Q^T ,$$

$$\nabla\varphi = h^{-1}Q(\nabla_q\varphi) ,$$

in which φ is any scalar field.

Variational principle in $\mathcal{Q}$

The change of variable $\mathbf{r} \to \mathbf{q}$ takes the physical exterior domain, $\mathcal{R}^e$, to a *proxy* domain, $\mathcal{Q}$. If one transforms the definition of F—*i.e.* the quantity to be minimized—from an integral over $\mathcal{R}^e$ to one over $\mathcal{Q}$ one gets

$$F := \iiint\limits_{\mathcal{Q}} \Big[h^{-3} \{\mathrm{div}_q\,(h\mathbf{U})\}^2 + (1/4) h^{-1} \big\| [\nabla_q\,(\mathbf{U}) - \nabla_q\,(\mathbf{U})^T] \big\|^2 \Big]\, dV_q \; ,$$

in which $\mathbf{U} := Q^T(h\mathbf{u})$.

Variational formalism in $\mathcal{Q}$, I

Proceeding as before, one may derive an identity satisfied by *the first variation*, δF, of F, namely

$$\delta F + \iint\limits_{\partial\mathcal{Q}} \delta\mathbf{U} \bullet [-\Gamma(\hat{\mathbf{n}}_q)]\, dA_q + \iiint\limits_{\mathcal{Q}} \delta\mathbf{U} \bullet (\mathrm{div}_q\,\Gamma - \mathbf{f})\, dV_q = 0\ ,$$

in which

$$\Gamma := 2h^{-2}\mathrm{div}\,(h\mathbf{U})I + h^{-1}\left[\nabla_q\,\mathbf{U} - (\nabla_q\,\mathbf{U})^T\right]\ ,$$

$$\mathbf{f} := 2h^{-3}\mathrm{div}\,(h\mathbf{U})\nabla_q\,h\ .$$

Variational formalism in $\mathcal{Q}$, II

If δF vanishes (*i.e.* F is stationary) for arbitrary variations, $\delta\mathbf{U}$ in $\mathcal{Q}$ and arbitrary tangential components of $\delta\mathbf{U}$ on $\partial\mathcal{Q}$ one deduces the *natural boundary condition* and Euler-Lagrange equation, respectively, namely

$$-\Gamma(\hat{\mathbf{n}}_q) = \mathbf{0} \quad , \quad \operatorname{div}_q \Gamma = \mathbf{f} \, ,$$

Boundary conditions for the normal velocity in $\mathcal{R}^i$ and $\mathcal{R}^e$

If $\mathbf{u}$ in $\mathcal{R}^i$ is referred to a frame in which the boundary sphere is at rest then $\mathbf{u} \bullet \hat{\mathbf{n}} = 0$ on $\partial\mathcal{R}^i$. If the parameter, A, is given then the equations given thus far determine $\mathbf{u}$ in $\mathcal{R}^i$ uniquely.

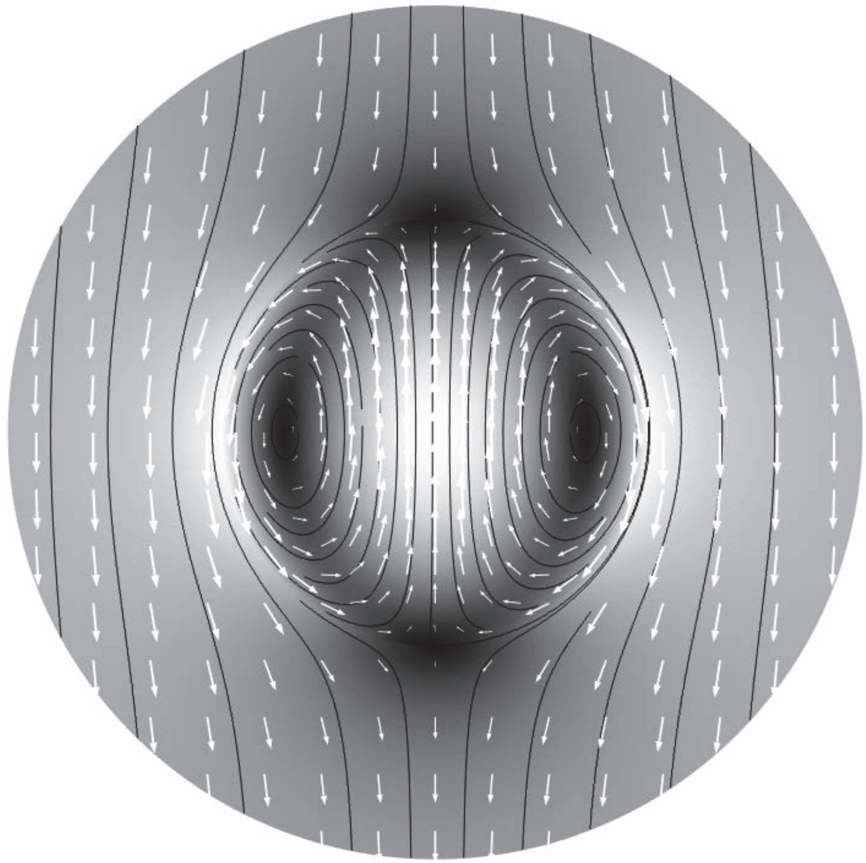
If $\mathbf{u}$ in $\mathcal{R}^e$ is referred to a frame in which the remote undisturbed fluid is at rest then $\mathbf{u} \bullet \hat{\mathbf{n}} = w_s \hat{\mathbf{i}}_3 \bullet \hat{\mathbf{n}}$ on $\partial\mathcal{R}^e$, in which w_s is the vertical velocity of the bounding sphere relative to that remote fluid. If w_s is given then the equations given thus far determine $\mathbf{u}$ in $\mathcal{R}^e$ uniquely.

Condition for the absence of a slip layer at the bounding sphere, I

If A and w_s are both given arbitrarily the component of $\mathbf{u}$ in the tangential, non-swirl direction is not, in general, the same on $\partial\mathcal{R}^i$ and $\partial\mathcal{R}^e$ (even after if $\mathbf{u}$ is referred to a common reference frame). To avoid a non-physical difference in pressure across the bounding sphere, one must determine either one of the parameters A or w_s in terms of the other.

Condition for the absence of a slip layer at the bounding sphere, II

In the present model w_s was given and the solutions for $\mathbf{u}$ in the $\mathcal{R}^e$ and $\mathcal{R}^i$ were calculated in Steps 1 and 2, respectively, of a *solver sequence*. In Step 2 A was computed by means of a Global ODE and DAE node to ensure that the circumferentially averaged northern component of fluid speed at the equator on $\partial\mathcal{R}^i$ agreed with the corresponding value obtained in Step 1.



Velocity in a plane containing the axis of symmetry: shading shows fluid speed; solid lines show streamlines; and arrows show velocity vectors.

Conclusions

Kelvin Inversion defines an orthogonal curvilinear coordinate system such that:

C1. A single scale factor is common to all three coordinates.

C2. Its associated system of unit vectors is left handed for true Kelvin Inversion but right handed for modified Kelvin Inversion;

C3. It maps an unbounded physical domain to a bounded proxy domain and enables the solution for the velocity in the latter.