Single Phase Flow Models In Fractal Porous Media Using A Fractional Continuum Mechanics Approach

Juan E. Linares-Pérez[1*], Martin A. Diaz-Viera[2]
[1]Instituto de Geofísica, (UNAM), [2]Instituto Mexicano del Petróleo (IMP)
*Corresponding author: Circuito de la Investigación Científica s/n, Ciudad Universitaria, 04510, México D. F., eduardo_linares@comunidad.unam.mx

Abstract: In this work, we present two models of single phase flow in porous media with fractal properties which were implemented using COMSOL Multiphysics® to simulate, analyze and interpret pressure tests in naturally fractured reservoirs. The models were derived using a systematic fractional continuum mechanics approach for isotropic and anisotropic fractal media, respectively.

One of the advantages of derived mathematical models is that they are represented in terms of conventional differential equations. Numerical experiments showed a consistent behavior with the expected anomalous behavior, where the pressure drops at a faster or slower rate compared to conventional flow model.

Keywords: single phase fluid flow, porous media, fractal continuum mechanics.

1. Introduction

The primary motivation of this work was to develop mathematical and numerical models for fluid flow in porous media with fractal properties, because it has been observed that the pressure from well tests of certain naturally fractured reservoirs in México exhibit an abnormal behavior which departs from the expected when traditional flow models with Euclidean geometry are applied.

Some authors like (Camacho-Velázquez et. al. 2006), (Barker 1988), (Chang and Yortsos 1990), among many others, believe that the reason for this anomalous behavior can be explained assuming that the porous medium has fractal properties due to the complex distribution of fractures. They have developed some fractal flow models; however the derivation of such models was not very rigorous because they only made some modifications to the conventional flow model to take into account some fractal characteristics of the medium.

Recently, (Tarasov 2005) and (Ostoja-Starzewski, et. al. 2011 and 2013) have introduced fractional measures for isotropic and anisotropic fractal media, respectively, which allowed for a systematic derivation of fractal flow models using a fractional continuum mechanics approach.

The theory of fractional continuum mechanics can be interpreted as a generalization of the usual theory of continuum mechanics but introducing fractional measures instead of a Lebesgue measure.

In this work, two models for single phase flow in porous media with fractal properties were derived to evaluate their performance numerically.

One of the most important aspects to note is that all models were developed following a systematic methodological approach which consists of four stages. The first step is defining a *conceptual model* where the assumptions, the scope and limitations of the model are established. In the second stage the *mathematical model* is derived, consisting on a partial differential equation with initial and boundary conditions that satisfy the assumptions of the conceptual model. While in the third stage is chosen the *numerical model* which is always a discretized version of the mathematical model. In this case a finite element method was applied. Finally, in the fourth stage the *computational model*, which is the implementation of the numerical method on a computer platform.

One of the advantages of the resulting mathematical models of anomalous flow obtained in this work is that they are represented in terms of conventional differential equations in which their coefficients are functions of the fractal (mass and boundary) dimensions, i. e., fractional differential equations can be expressed as equations with integer derivatives, which has a great advantage for their numerical solution and especially for its computational implementation.

Numerical simulations were carried out for each one of the two models, for different values of fractal dimensions in a case study in a square domain in two dimensions.

Numerical results showed consistency with the expected anomalous behavior, where the pressure drops at a faster or slower rate compared to the conventional flow model.

2. Conceptual Model

The following assumptions are usually adopted for models of fluid flow through porous media (Chen et al. 2006):

- There are two phases: a fluid phase and a solid phase.
- There are only two components: the fluid phase consists of one fluid component and the solid phase is made of the rock, which usually is called the porous matrix.
- The porous medium is fully saturated with the fluid.
- The solid matrix remains at rest throughout the fluid-flow process.
- The rock and fluid are slightly compressible. This means that the fluid density and the rock porosity can be approximated as linear functions of the pressure.
- The fluid velocity fulfills Darcy's law.
- The fluid is not subjected to diffusion processes.
- The fluid viscosity is constant.
- The system is under isothermal conditions.

Additionally, it is assumed that the porous medium has fractal properties, and that could be isotropic or anisotropic.

The flow system is a two-phase system since it consists of the solid matrix and the fluid contained in its pores. However, the fact that the motion of the solid phase is known, since it remains still, permits dealing with the fluid phase exclusively and treating the system as a single-phase system. This single phase, in turn, is made of only one component, the fluid. Thus, the family of extensive properties consists of only one extensive property, namely, the fluid mass.

3. Mathematical Model

The governing equations for the single phase flow of a fluid (a single component or a homogeneous mixture) in a porous medium are given by a mass balance equation, a momentum conservation equation (Darcy's law) and an equation of state. We make the assumptions that the mass fluxes due to dispersion and diffusion are so small (relative to the advective mass flux) that they are negligible and that the fluid-solid interface is a material surface with respect to the fluid mass so that no mass of this fluid can cross it. Denote by ϕ the porosity of the porous medium, by ρ the fluid mass density per volume unit and by $\underline{u}$ the Darcy velocity.

3.1. Isotropic Fractal Model

Applying the theory of isotropic fractional continuum mechanics (Tarasov 2005), we define the extensive property as the fluid mass in a fractal homogeneous medium M_D and the intensive property as the product of density ρ and the porosity ϕ in the fractal media.

$$M_D = \int_{B(t)} \phi(\underline{x},t)\rho(\underline{x},t)d\mu_D \tag{1}$$

Where

$$d\mu_D = \frac{2^{2-D} r^{D-2}}{\Gamma(D/2)} d\mu_2 \tag{2}$$

is a fractional measure introduced by Tarasov (Tarasov 2005) for isotropic fractal media, $d\mu_2$ is the Lebesgue measure in $\mathbb{R}^2$, $r = |\underline{x} - \underline{x}_0|$ where $\underline{x}_0 \in B(t)$ is the initial point of the fractional integral in the Rietz form, and Γ is the Gamma function. Due to there is conservation of fluid mass, the global balance equation of fluid mass in isotropic fractal porous media can be expressed as follows

$$\frac{d}{dx} M_D(t) = 0 \tag{3}$$

and the local balance fractional differential equation is given by

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla^D \cdot (\rho\phi\underline{v}) = 0 \tag{4}$$

Now, the fluid velocity in the porous media is defined as $\underline{u} = \phi\underline{v}$ then

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla^D \cdot (\rho \underline{u}) = 0 \quad (5)$$

Considering the rock and the fluid slightly compressible (Chen et al. 2006), the time derivative term can be expressed as

$$\frac{\partial}{\partial t}(\rho\phi) = \phi c_t \frac{\partial p}{\partial t} \quad (6)$$

where p is the pressure and $c_t = c_R + c_f$ is the total compressibility, whereas c_R and c_f are the rock and fluid compressibility, respectively. The Darcy law for isotropic fractal media (Linares and Díaz-Viera 2014) can be expressed as:

$$\underline{u} = -\frac{1}{\mu}\underline{\underline{k}} \cdot \left(\nabla^D p + \rho\gamma\nabla^D z\right) \quad (7)$$

Where μ is the fluid viscosity, $\underline{\underline{k}}$ is the permeability tensor of porous media, p is the fluid pressure, ρ is the fluid density, γ is the gravitational acceleration and z is the high.

Here the fractional gradient operator is defined as

$$\nabla^D f \equiv c_2^{-1}(D,r)\nabla(c_1(d,r) f) \quad (8)$$

where

$$c_2(D,r) = \frac{2^{D-2} r^{2-D}}{\Gamma(D/2)} \;, \quad (9)$$

$$c_2^{-1}(D,r) = \frac{1}{c_2(D,r)} \quad (10)$$

and

$$c_1(d,r) = 2^{1-d}\frac{\sqrt{\pi} r^{d-1}}{\Gamma(d/2)} \quad (11)$$

Note that the Darcy law for isotropic fractal media Eq. (7) was derived based on the work of (Neumann 1977) and using the fractional measure Eq. (2) introduced by Tarasov.

Substituting Eqs. (7) and (6) in Eq. (5), and ignoring the gravity effect, a single phase flow model for isotropic fractal porous media is obtained:

$$\rho\phi c_t \frac{\partial p}{\partial t} = \nabla^D \cdot \left(\frac{\rho}{\mu}\underline{\underline{k}} \cdot \nabla^D p\right) \quad (12)$$

If the change of density is neglected then the fractional differential equation (12) can be expressed as:

$$\phi c_t \frac{\partial p}{\partial t} - \nabla^D \cdot \left(\frac{1}{\mu}\underline{\underline{k}} \cdot \nabla^D p\right) = 0 \quad (13)$$

3.2. Anisotropic Fractal Model

In the same form, applying the fractional continuum mechanics theory to anisotropic media (Ostoja-Starzewski, et. al. 2011, 2013), we define the extensive property as the mass fluid of the homogeneous fractal media, as $\tilde{M}_D$, as:

$$\tilde{M}_D = \int_{B(t)} \phi(\underline{x},t)\rho(\underline{x},t)\, d\tilde{\mu}_D \quad (14)$$

Where

$$d\tilde{\mu}_D = dl_{\alpha 1}(x_1)\, dl_{\alpha 2}(x_2) \quad (15)$$

while the length measurement along each coordinate is given by the transform coefficients:

$$dl_{\alpha k}(x_k) = c_1^{(k)}(\alpha_k, x_k)\, dx_k \quad (16)$$

with $D = \alpha_1 + \alpha_2$, and

$$c_1^{(k)} = \frac{|x_k|^{\alpha_k - 1}}{\Gamma(\alpha_k)}, \quad k = 1, 2 \text{ (no sum)}. \quad (17)$$

Again, considering fluid mass conservation

$$\frac{d}{dx}\tilde{M}_D(t) = 0 \quad (18)$$

Then, the local balance differential equation is

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho\phi\underline{v}) = 0 \quad (19)$$

Applying the fluid velocity in the porous media $\underline{u} = \phi\underline{v}$ then we have

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho\underline{u}) = 0 \quad (20)$$

Considering density a linear approximations of the pressure, as in the equation (21) results

$$\frac{\partial}{\partial t}(\rho\phi) = \phi c_t \frac{\partial p}{\partial t} \quad (22)$$

The Darcy law for anisotropic fractal media (Linares and Díaz-Viera 2014) is expressed as:

$$\underline{u} = -\frac{1}{\mu}\underline{\underline{k}} \cdot \left(\tilde{\nabla}^D p + \rho\gamma\nabla z\right) \quad (23)$$

Here the fractional gradient operator is defined as

$$\tilde{\nabla}^D f \equiv \left(\frac{1}{c_1^{(1)}}\frac{\partial f}{\partial x_1}, \frac{1}{c_1^{(2)}}\frac{\partial f}{\partial x_2}\right) \quad (24)$$

The rest of the notations have the same meaning as in equation (7). Substituting the equations (22) and (25) in equation (20) and neglecting the gravity effect and the change of density, we have a single phase model of anisotropic fractal flow:

$$\phi c_t \frac{\partial p}{\partial t} = \nabla \cdot \left(\frac{1}{\mu} \underline{\underline{k}} \cdot \tilde{\nabla}^D p \right) \qquad (26)$$

3.3. Initial and Boundary Conditions

Initial conditions:

$$p(t_0) = p_0 \qquad (27)$$

Boundary conditions:

No-flow conditions at all boundaries.

$$\underline{u} \cdot \underline{n} = 0 \qquad (28)$$

4. Numerical and Computational Models

For the numerical solution we apply the following methods:

- A backward finite difference discretization of second order for the temporal derivatives was used resulting a full implicit scheme in time.
- A standard finite element discretization with quadratic Lagrange polynomials as weighting and base functions.
- An unstructured mesh with triangular elements in 2D.
- A variant of the LU direct method for non-symmetric and sparse matrices, implemented in the UMFPACK library, for the solution of the resulting algebraic system of equations.

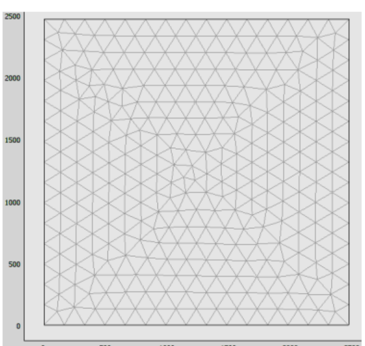

Figure 1. Square domain with triangular element mesh.

The computational implementation was carried out for a rectangular domain in two dimensions (see Figure 1) using the COMSOL Multiphysics software by the PDE mode in general coefficient form for the time-dependent analysis (COMSOL Multiphysics 2007).

4.1. Isotropic Model

Considering the equation (13) for a two dimensional region, we develop the fractional divergence in $\mathbb{R}^2$ and multiply both sides by $c_2(D,r)$ to obtain the following differential equation with a conventional divergence

$$c_2(D,r)\phi c_t \frac{\partial p}{\partial t} \qquad (29)$$
$$-\nabla \cdot \left(c_2^{-1}(D,r) c_1(d,r) \frac{1}{\mu} \underline{\underline{k}} \cdot \nabla (c_1(d,r) p) \right) = 0$$

Now, developing the gradient results

$$c_2(D,r)\phi c_t \frac{\partial p}{\partial t} \qquad (30)$$
$$-\nabla \cdot \left(\zeta \frac{1}{\mu} \underline{\underline{k}} \cdot \nabla p + \zeta \frac{1}{\mu} \underline{\underline{k}} (d-1) r^{-2} \underline{x} p \right) = 0$$

where $\zeta := c_2^{-1}(D,r) c_1^2(d,r)$.

Using the template of COMSOL PDE mode for time dependent analysis in the coefficient form:

$$e_a \frac{\partial^2 p}{\partial t^2} + d_a \frac{\partial p}{\partial t} \qquad (31)$$
$$-\nabla \cdot (c \nabla p + \alpha p - \gamma) + \beta \cdot \nabla p + ap = f$$

where:

$d_a = c_2(D,r)\phi c_t$; $c = \zeta \frac{1}{\mu} \underline{\underline{k}}$; $\alpha = \zeta \frac{1}{\mu} \underline{\underline{k}} (d-1) r^{-2} \underline{x}$

with $\underline{x} = (x - x_0, y - y_0)$

while $e_a = \gamma = \beta = a = f = 0$.

4.2. Anisotropic Model

Considering the equation (26) in COMSOL notation (PDE, Coefficient Form) Eq. (31), where: $e_a = \alpha = \gamma = \beta = a = f = 0$ while,

$d_a = \phi c_t$, and $c = \frac{1}{\mu} \tilde{\underline{\underline{k}}}$.

The permeability tensor can be defined as:

$$\underline{\underline{k}} = \begin{pmatrix} k_{11} / c_1^{(1)} & 0 \\ 0 & k_{22} / c_1^{(2)} \end{pmatrix} \qquad (32)$$

For boundary conditions in both models is used a generalized form of the Neumann boundary condition implemented in COMSOL which are expressed in the next form:

$$n \cdot (c \nabla p + \alpha p - \gamma) + qp = g \qquad (33)$$

where $\alpha = \gamma = q = g = 0$.

5. Numerical Simulations

Basically, numerical simulations consist in solving the flow models derived in section 3 for a case study defined in a square domain with a production well at the center for different combinations of fractal dimensions using data taken from (Chen et al. 2006), see Table 1.

Table 1: Data for the case study.

Symbol	Description (value)
p_0	Initial pressure (3600 psi)
μ	Oil viscosity (1.06 cP)
k	Permeability (0.3 Darcy)
x_0	x-coordinate of the well (1234.44 m)
y_0	y-coordinate of the well (1234.44 m)
c_f	Oil compressibility (0.00001 1/psi)
c_R	Rock compressibility (0.000004 1/psi)
c_t	Total compressibility (0.000014 1/psi)
ϕ	Porosity (0.2)
Q_0	Oil production rate (300 STB/D)

6. Discussion of Results

In Figure 2, it can be observed that the pressure in the well drops faster at the beginning but quickly stabilizes in a constant value with the increasing of the fractal boundary dimension for a fixed value of mass fractal dimension ($D = 2$). Note that the conventional model ($D = 2$ and $d = 1$) is in red color. While in Figure 3, it is seen that the pressure behavior tends to be more linear as we move away from the well with the increasing of the fractal boundary dimension.

Figure 4 shows that with the decrease of the mass fractal dimension (D) for a fixed fractal boundary value ($d = 1$) in the isotropic fractal model the behavior of the pressure drop in the well at the beginning is slower but later is faster becoming almost linear, which is very different in comparison with the conventional model ($D = 2$ and $d = 1$) in red color.

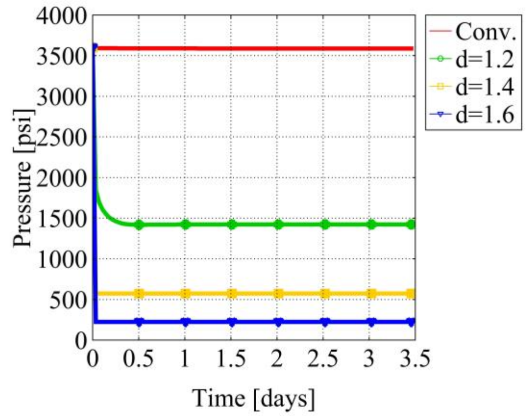

Figure 2 Pressure drop in the well during 3.5 days, for the isotropic fractal model with $D = 2$ and different values of the boundary dimension (d).

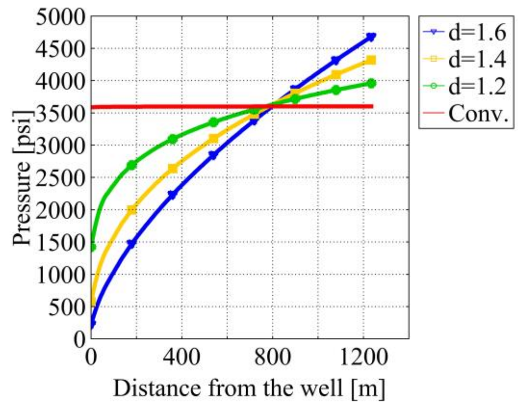

Figure 3 Pressure profile along a section for the isotropic fractal model with $D = 2$ and different values of the fractal boundary dimension (d).

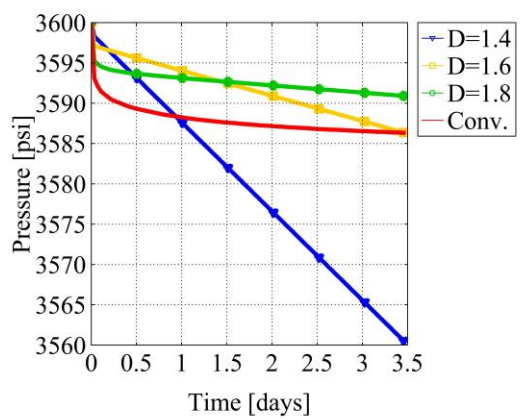

Figure 4 Pressure drop in the well during 3.5 days for the isotropic fractal model with $d = 1$ and different values of mass fractal dimension (D).

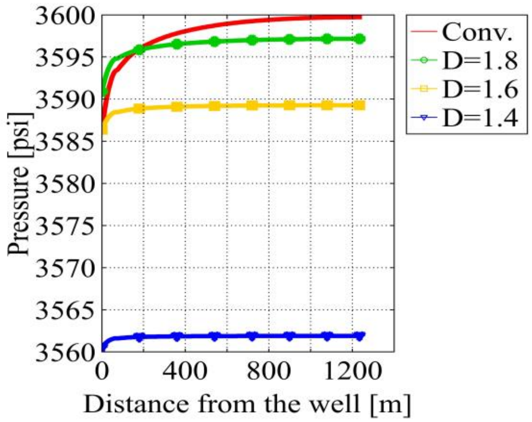

Figure 5 Pressure profile along a section for the isotropic fractal model with $d = 1$ and different values of mass fractal dimension (D).

While in Figure 5, it is seen that the pressure behavior tends to be lower but quickly stabilizes in a constant value as we move away from the well with the decreasing of the mass fractal dimension.

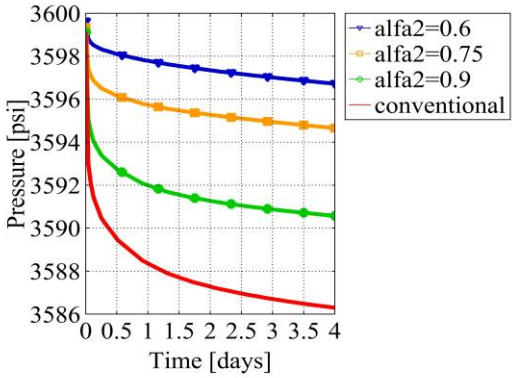

Figure 6 Pressure in the well during 4 days, with $\alpha_1 = 1$ and different values of α_2 for the anisotropic fractal model.

In the anisotropic model (Figures 6 and 7), the behavior of the pressure drop in the well is similar but slower as the mass fractal dimension $D = \alpha_1 + \alpha_2$ increases in comparison with the conventional flow model. This can be interpreted as the connectivity of the medium decreases as the value of the mass fractal dimension increases.

The behavior of the pressure around the well is symmetric if the medium is isotropic $\alpha_1 = \alpha_2$ (see Figure 8) and asymmetric if the fractal medium is anisotropic $\alpha_1 \neq \alpha_2$ (see Figure 9) when the anisotropic fractal model is applied.

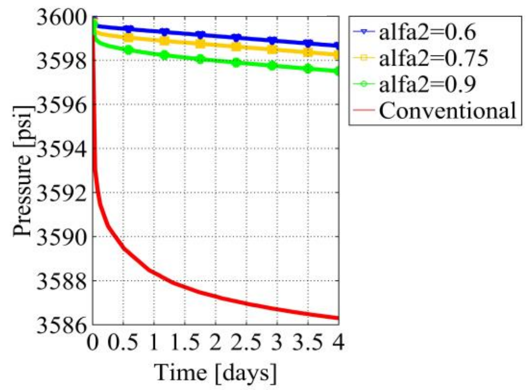

Figure 7 Pressure in the well during 4 days, with $\alpha_1 = 0.6$ and different values of α_2 for the anisotropic fractal model.

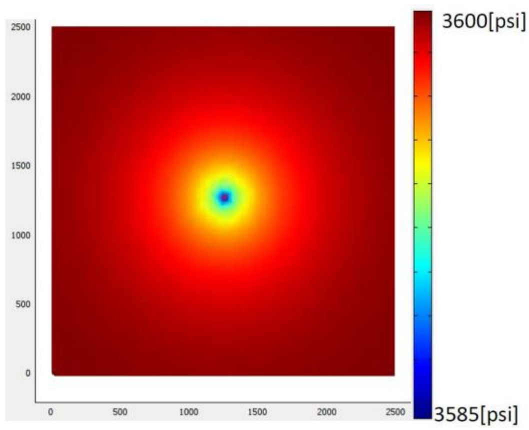

Figure 8 Pressure in 4 days for the anisotropic fractal model with $\alpha_1 = 0.6$ and $\alpha_2 = 0.6$.

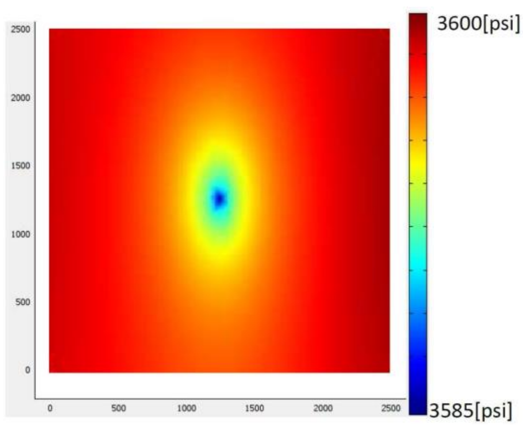

Figure 9 Pressure in 4 days for the anisotropic fractal model with $\alpha_1 = 0.75$ and $\alpha_2 = 0.6$.

7. Conclusions

Applying a fractional continuum mechanics approach two single phase flow models in porous media with fractal properties were derived. The first one was developed for isotropic fractal media using a fractional measure introduced by (Tarasov 2005). Whereas the second model was obtained applying a fractional measure introduced by (Ostoja-Starzewski, et. al. 2011 and 2013) for anisotropic media. Both models required unconventional Darcy laws for fractal media that were derived from the balance equation of linear momentum using the aforementioned fractional measures following the same approach given in (Neumann 1977).

The numerical experiments showed a behavior consistent with the question of anomalous diffusion, where the pressure drops with faster or slower rate compared to the conventional flow model.

One of the advantages of the mathematical models of abnormal flow obtained in the present work is that they are conventional differential equations with additional numerical coefficients, i.e., fractional differential equations can be expressed in terms of integer derivatives, the latter being a great advantage for their numerical solution and computational implementation.

The solutions of fractal flow models are reduced to the solution of the conventional model if the corresponding integer dimensions ($D=2$ and $d=1$ or $\alpha_1=1=\alpha_2$) are taken.

Comparing the isotropic model with respect to the anisotropic model it can be seen that the first one in general depends on three parameters (D, d, $\underline{x}_0$), but strongly depends in particular on the choice of the point location ($\underline{x}_0$) which is usually placed in the same position of the source term, while the second model only depends on the fractal dimensions in each direction (α_1,α_2). Moreover, the anisotropic model despite of being more general doesn't reduce to the first one. Therefore, both models constitute two alternatives for modeling flow in fractal porous media.

As future work, the developed methodology can be extended to multiphase flow and multicomponent transport models in porous media with fractal properties, with great potential for application in heterogeneous reservoirs modeling.

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8. Acknowledgements

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